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On Mathematical Certainty

Within the body of mathematical science nearly all the concepts and principles used are purposely invested with meanings unique for all minds. Yet *mathematics* itself, the grand ensemble of these concepts and principles, taken with their logical implications, has not a universally accepted meaning. Analysis of many of the definitions of mathematics—some of them now classic—reveals important differences. Compare Peirce's "the science of necessary conclusions," W. Benjamin Smith's "the universal art apodictic," Russel's "the science, of the truth or the falsity of whose propositions we are ignorant," and other definitions more or less familiar to everyone. They carry implications so lacking in uniformity that, conceivably, a sense, not of scientific consistency but of confusion may easily be experienced by an inquirer brought for the first time face-to-face with such non-uniformity

We are not here concerned to account for this diversity of definitions of mathematics, though it is probable that, (a) the vast polyphasedness of the science, (b) the growth and multiplication of its branches and applications throughout many centuries have largely contributed to bring it about. Nor would we be bold to undertake to distill from this diversity a single definition worthy of universal adoption.

Our present aim is simply to bring into clearest possible view one or more *characteristics* of processes generally admitted to be mathematical in character. For example, one characteristic of a mathematical process is the *certainty* of its *deductions*. Another is the *uniqueness* of its *conclusions*. A third is its inclusion at times of *order* or *number* concepts, or both.

In this issue of the Magazine we write only on the nature of what is called *Mathematical Certainty*. Obviously the existence of mathematical certainty implies the existence of other kinds of certainty. Thus, there is the certainty that I hold a pencil and am writing with it. There is the certainty that Roosevelt, not Hitler, is President of

the United States. The first certainty is a conscious one, the second is of a somewhat different kind. If in a vivid dream I fly to the top of a tree, my consciousness of doing so is a third sort of certainty, a certainty only in relation to my dream. In a sense every kind of certainty is only relative. For, our personal existence, including our states of mind, can *only* be assumed, not proven with *absoluteness*. Thus it seems natural to assume the existence of some condition for the generation of each kind of certainty. What are the special conditions that generate mathematical certainty? As a first step in answering this question, we set up some examples of mathematical certainty universally granted to be examples. We then examine these to determine the sources or conditions generating the certainty.

The following are some examples of universally accepted mathematical certainties. By "universally accepted" is meant "accepted by all familiar with ordinary mathematical symbols and operations." The phrase is used in precisely the same sense as would be used such words as *universally accepted tennis rules*, a language that leaves out of account one ignorant of tennis.

(a) $5 \cdot 7 = 7 \cdot 5 = 35$.

(b) $7^3 \cdot 8^3 = 343 \cdot 512 = (7 \cdot 8)^3 = 56^3 = 56 \cdot 56 \cdot 56 = 175616$.

(c) $3^2 \cdot 3^5 = 3^7 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$.

These are samples of what may be called simple mathematical certainties. Only very slight reflection is needed to discover the conditions which generate these certainties. Tabulated with complete detail, they are:

(1) Every symbol employed such as 7, 8, 3, 343 has one and just one meaning, or value.

(2) This meaning is universally accepted.

(3) Every operational symbol such as " = ", or the dot in 7.3, or the combination of an upper and a base symbol, as 3^7, has one and only one meaning.

(4) This meaning is universally accepted.

(5) Symbols, or marks, having ambiguous or multiple meanings are wholly absent from the chain of symbols characterizing a simple mathematical certainty.

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For the mind ignorant of, or not accepting, the meaning assigned to these symbols of value and of operation there is no mathematical

certainty. Similarly for one ignorant of the differential calculus there is no mathematical certainty that the derivative with respect to "t" of t^3+t^2-1 is $3t^2+2t$. But after he has (a) ascertained the accepted meaning of a *variable*, and of a *continuous function* of the variable, and has (b) ascertained and accepted the *character* and order of the operations generally adopted as defining a unique result called the *derivative*, then, *not until then*, is he *mathematically certain* that the derivative with respect to "t" of t^3+t^2-1 is $3t^2+2t$.

Against any charge that we are needlessly elaborating something that is already obvious, it is sufficient to remind the reader that until mathematics has been defined acceptably to all minds, mathematical certainty is hardly to be described as something obvious.

It may be assumed that the more complicated instances of mathematical certainty are but concatenations or complexes of the simpler ones in combination with the certainties of logical inference. Consider, for example, the chain of symbols of value, of operations and necessary implications which terminate in the *unique* conclusion: *that an equation of the nth degree with coefficients in the field of complex numbers has a root. The certainty of this conclusion for every mind is generated by the following conditions:*

(1) It is *agreed* that $a+bi$ in which a and b are any real numbers and $i=\sqrt{-1}$, shall be called a *complex* number.

(2) It is *agreed* that a one-to-one correspondence may be established between all complex numbers and all the points of a plane.

(3) It is *agreed* that $P(z)=w=a_0z^n+a_1z^{n-1}+\cdots+a_n$ is a polynomial in the complex variable $x+yi=z$.

(4) It is *agreed* that if $P(z_0)=0$, z_0 is a root of the equation $P(z)=0$.

(5) *Agreed* that $|z|=\sqrt{x^2+y^2}=$modulus of z.

(6) *Agreed* that $(a+bi)(c+di)=(ac-bd)+(bc+ad)i$ and that $(a+bi)+(c+di)=(a+c)+(b+d)i$.

(7) Agreement (6) *implies* the *certainty* that $P(z)$, or $w=$a complex variable.

(8) By agreement (5) and by (7) it follows with certainty that $w=P(z)$ has a modulus.

(9) It may be inferred that, corresponding to some value z_0 of z is a minimum value $|w_0|$ of $|w|$.

(10) Obviously this minimum value of $|w|$ is either zero or a real number not zero.

(11) If the minimum value of $|w|$ is not zero, call it p.

(12) But from the supposition that a minimum value of $|w|$ is different from zero may be inferred with certainty that $|w|$ has a smaller value than p, a contradiction of the assumption that p (not equal to zero) is a minimum $|w|$.

(13) Thus the minimum $|w|$ is zero.

(14) $\therefore z_0$ is a root of $|w| = 0$.

By now it should be clear that the certainty of a mathematical conclusion arises from induced states of mind similar in all individuals, all minds being essentially alike in structure and function. Items (1)-(6) in the above classification are but agreements or assumptions which the would-be demonstrator of the truth of the theorem proposed calls on the reader to accept, *understandingly*. The psychology of this acceptance is in nearly all respects that of one who, on undertaking to play a game, accepts and abides by the rules which define the game.

Items (7)-(13) correspond to states of mental conviction arising from, (a), the clear perception of unique consequences of machine-like operations performed upon granted conditions, assumptions, or definitions, (b), the clear perception of some necessary implication, or implications, of assumptions, definitions, or of previous implications. An instance of necessary implications is item (14). The certitude of mind generated by its use proceeds from one's assumption that $|w|$ is either zero or not zero. So, if being not zero is shown to be contrary to both agreements and products of accepted operational machinery, then it is *certain* that $|w| = 0$.

If I toss a coin in the air, no condition exists that will generate a *certainty* that it will fall "head up." But if it is so thrown that "head up" and "tail up" are equally likely to happen, a measure of the probability that it will fall "head up" is $\frac{1}{2}$. The condition generating the mathematical *certainty* that this probability is $\frac{1}{2}$ is the Definition of Probability Measure.

If one accepts the agreements of Euclidean geometry, he may become *mathematically certain* that the sum of the angles of a plane triangle is 180°. But if he puts aside Euclidean geometry and plays one of the non-Euclidean geometry games, he is led to be *mathemati-*

cally certain that the sum of the angles of a plane triangle is not 180° but more or less than 180°.

In closing this study we leave the following questions with the reader:

(1) Is one who is acquainted with the Latin language mathematically certain that "arma virumque cano" = "I sing of arms and a man" ?

(2) If a coin is thrown into the air from the top of a tall tower, is it mathematically certain to reach a maximum height?

S. T. SANDERS.

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Fundamental Distribution of Errors for Agricultural Field Trials

By G. A. BAKER
University of California

1. *Introduction.* The analysis of variance technique as developed by Professor R. A. Fisher and others has been widely applied in the analyses of field trial data both to yield data and attack rate data. Two fundamental difficulties with the application of the analysis of variance technique as applied to Randomized Blocks and Latin Squares has been pointed out by Neyman[1] and McCarthy[2] and others. These difficulties deal with two assumptions made in the derivation of the analysis of variance technique: (a) The fertility level of each plot or potential producing ability assigned to each variety or treatment is independent of fertility levels of the plots assigned to any other variety or treatment. This can not be true for if the best plot is assigned to one variety then a worse plot must be assigned to another so that a negative correlation exists. (b) The variances are equal for all varieties or treatments under consideration. The difficulty in connection with this assumption as applied to wheat bunt experiments and in general has been pointed out by Salmon.[3]

McCarthy's work shows that the analysis of variance technique tends to over-estimate significance. Salmon's work shows that the straight forward application of analysis of variance to attack rate data is of very doubtful validity.

There is one remaining assumption at the basis of analysis of variance technique applied to field trial data which should be more thoroughly investigated. That is the assumption that the random errors are normally distributed. It is the purpose of this paper to investigate the fundamental distribution of random errors in field trial data and attack rate data, and indicate some of the implications of the investigation.

2. *Fundamental distribution of random errors in field trial data.* That the random errors are normally distributed is made plausible by data on the yields of wheat on 500 plots each of 1/500 acre area presented by Mercer and Hall.[4] Similar data have been collected by Wiebe[2] whose plots of wheat contained 15 square feet instead of the 87 plus square feet of the Mercer-Hall plots. A grouped frequency

distribution of Wiebe's plot yields is presented in table one with constants calculated from the ungrouped data. The distribution is significantly non-normal being slightly skewed to the left because of the one very extreme item of 75 grams and is nearly rectangular at the center and has heavier tails than a normal distribution. Small deviations are less probable than for a normal distribution but deviations of a moderate and very large size are more probable.

TABLE I

GROUPED FREQUENCY DISTRIBUTIONS AND CONSTANTS OF 1500 YIELDS OF WHEAT IN GRAMS ON NURSERY PLOTS OF 15 SQUARE FEET. CONSTANTS CALCULATED FOR UNGROUPED DATA. THE THEORETICAL NORMAL FREQUENCIES ARE GIVEN FOR COMPARISON

Class Interval	*Number*	*Theoretical Frequency*	*Class Interval*	*Number*	*Theoretical Frequency*
62- 91.....	1		542-571.....	161	171.0
92-121.....	0		572-601.....	161	178.8
122-151.....	0	0.1	602-631.....	176	171.0
152-181.....	0		632-661.....	145	149.6
182-211.....	0		662-691.....	112	119.7
212-241.....	0	0.3	692-721.....	70	87.5
242-271.....	0	0.8	722-751.....	44	58.6
272-301.....	0	2.1	752-781.....	32	35.8
302-331.....	0	4.8	782-811.....	23	20.0
332-361.....	0	10.3	812-841.....	14	10.3
362-391.....	5	20.0	842-871.....	8	4.8
392-421.....	19	35.8	872-901.....	4	2.1
422-451.....	82	58.6	902-931.....	3	0.8
452-481.....	125	87.5	932-961.....	4	0.3
482-511.....	141	119.7	962-991.....	1	
512-541.....	168	149.6	992-1021....	1	0.1
				1500	1500.0

$\mu_1' = 587.26$ $\beta_1 = .057$ $g_1 = .24 \pm .06$*

$\mu_2 = 10{,}105.86$ $\beta_2 = 4.27$ $g_2 = 1.28 \pm .13$

$\mu_3 = -242{,}151.00$

$\mu_4 = 435{,}861.894$

If the data of Wiebe are combined by 5's making 300 plots of 75 square feet to compare with the Mercer-Hall plots the resulting distri-

*Standard errors are used throughout instead of probable errors.

bution is much nearer normal. A grouped frequency distribution of the yields of the 300 75-square foot plots with constants calculated for the ungrouped data are presented in table two. For comparison the normal curve frequencies are given. The constants g_1 and g_2 are not significantly different from zero but X^2 indicates that there is still a significant departure from normality. The indications are that the Mercer-Hall plots were too large to indicate the true distribution of the random errors. Plots less than 15 square feet should be considered. The data of tables one and two are so rough in some respects that representation by any smooth curve based on the first four moments, say, cannot be considered very satisfactory.

TABLE II

GROUPED AND THEORETICAL NORMAL FREQUENCY DISTRIBUTIONS OF 300 WHEAT YIELDS IN GRAMS OF 75-SQUARE FOOT PLOTS AND CONSTANTS CALCULATED WITHOUT GROUPING

Class Interval	*Observed Frequencies*	*Theoretical Normal Frequencies*
203-222	5	15.5
223-242	34	20.8
243-262	45	35.4
263-282	41	48.7
283-302	49	54.6
303-322	52	49.6
323-342	35	36.7
343-362	23	22.0
363-382	5	10.7
383-402	5	4.2
403-422	4	1.4
432-442	2	.4

$\mu_1' = 293.62$

$\mu_2 = 1{,}918.60$

$\mu_3 = 47{,}572.59 \qquad \beta_1 = .32 \qquad g_1 = .6 \pm .4$

$\mu_4 = 12{,}263{,}888 \qquad \beta_2 = 3.33 \qquad g_2 = .4 \pm .3$

$X^2 = 33.1$

$P = .0002$

NOTE: The intervals 403-422 and 423-442 were combined in calculating chi-square.

Further light may be shed on the distribution of random errors by a consideration of the distributions of the measurements of physical, chemical and bacteriological characteristics of the soil which are known to affect the yield of crops. It may be that a preliminary mapping of some of these measurements would permit plots to be selected in such a way as to equalize the fertility levels of the various sets of plots but that mere inspection will not be sufficient to do this is shown by the following data compiled from various sources.

Waynick[6] and Waynick and Sharp[7] have made studies of nitrification and nitrogen and carbon on plots especially selected for their extreme uniformity as nearly as could be judged by inspection. The grouped frequency distributions and constants calculated without grouping for various nitrogen and carbon determinations for systematic coverage of carefully selected plots of small size are given in Table III.

If the measurements on nitrogen, carbon, etc., are distributed in any way, i. e., are not all the same on uniform plots of ground it is indicated that the distribution of the random errors cannot be normal. The distribution of measurements on nitrogen, carbon, etc. may indicate the nature of the distribution of fertility, i. e., the range and density of the aggregate of the values of the fertility. If the actual range and density of the values of the fertility levels and the spread of the estimates of yield about a point were known then the distribution of the random errors would be known. It is for this reason that measurements in detail on the properties and constituents of plots of land and the correlation of these measurements with yields recorded for very small subdivisions is very desirable.

Table III shows that such fundamental soil quantities as nitrogen and carbon are distributed over a wide range, that these distributions may be approximately normal as for the Davis soil or very non-normal as for the Oakley soil and that nitrogen and carbon are highly correlated in their distribution. Nitrogen and nitrogen after 28 days incubation show a much smaller degree of correlation.

Veihmeyer and Hendrickson[8] have presented data showing that soil moisture and texture may vary widely and unpredictably within very short distances in soil that would be judged very uniform by ordinary methods of examination and that these factors have an important bearing upon root penetration and distribution.

That the bacterial and fungal population of the soil may vary very erratically from point to point in a small area of uniformly treated soil has been shown by Leach and Davey[9]. For example Table IV gives the distribution of Sclerotia per 400 gram sample of soil on a plot 300x300 feet, the samples being scattered uniformly over the plot.

TABLE III

COMPARISON OF GROUPED FREQUENCY DISTRIBUTIONS OF SYSTEMATIC NITROGEN AND CARBON DETERMINATIONS ON SMALL UNIFORM PLOTS WITH THEORETICAL NORMAL DISTRIBUTIONS. CONSTANTS ARE CALCULATED WITHOUT GROUPING (6) AND (7).

No. of Class interval	*Residual Nitrate*[1] *in soil, first 6″*		*Nitrate*[1] *after 28 days incubation*		*Oakley (Sand) Soil*				*Davis Soil (silty clay loam)*			
					Nitrogen[2]		*Carbon*[2]		*Nitrogen*[2]		*Carbon*[2]	
	Actual	*Theoretical*	*Actual*	*Theoretical*	*Actual*	*Theoretical*	*Actual*	*Theoretical*	*Actual*	*Theoretical*	*Actual*	*Theoretical*
1	4	2.2	4	2.1	9	17.6	1	1.8	3	1.4	1	.8
2	5	4.9	4	2.6	27	18.3	0	3.1	1	2.6	1	2.7
3	13	10.5	4	4.8	32	22.4	3	6.3	6	5.6	8	5.3
4	13	16.1	3	7.4	15	19.9	20	10.6	8	10.0	12	9.0
5	12	18.1	5	10.1	6	12.9	21	14.9	14	14.7	12	12.5
6	16	14.8	17	11.9	6	6.1	21	17.3	16	17.1	16	15.3
7	9	8.9	8	12.1	1	2.1	24	16.6	18	16.7	16	15.7
8	6	4.7	13	10.7	2	.5	7	13.1	16	13.7	11	15.1
9	3	1.6	9	8.2	0		5	8.6	5	9.0	7	10.8
10			8	5.4	1	.1	4	4.6	8	5.0	8	6.5
11			4	3.1	1		0	2.1	2	2.1	3	3.2
12			2	2.6			3	.8	2	1.1	1	1.3
13							0				2	.6
14							0	.3			1	.1
15							0					
16							1					
TOTAL	81	81.0	81	81.0	100	100.0	100	100.0	99	99.9	99	99.0
MEAN	2.70		3.40		.032		.440		.100		1.111	
σ	.70		1.06		.007		.115		.009		.100	
Correlation			.27				.8			.8		

[1] Milligrams of nitrate nitrogen per 100 grams of soil. [2] Percent of dry soil.

TABLE IV

SCLEROTIA PER 400 GRAMS OF SOIL, FIELD R, 1934
300x300 FEET

17	14	16	9	13
29	16	57	6	43
20	8	23	29	13
10	24	10	32	36
24	27	4	51	3

The data presented so far indicate that the ability to produce must be considered as a function of the point at which the seed is applied to the soil or of this point and the neighborhood that can be progressively penetrated by the growing plant's roots for a given seed spacing, weather conditions, etc.

Suppose a plot with a coordinate system as shown in Figure 1.

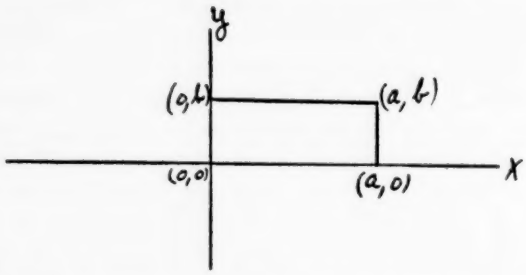

Figure 1. Rectangular plot with superimposed coordinate system.

Suppose further that the "intrinsic" yield at each point is $f(x,y)$ and that the observed yield at any point is normally distributed about the "intrinsic" yield with a constant standard deviation σ. It is conceivable that σ is also a function of x and y or position, that the distribution is not normal about the "intrinsic" yield at the point (x,y) and that f depends on time at least for some crops. If f depends on time, the integral (2.1) is a triple integral. The function $f(x,y)$ may not be continuous and if not, the integrals considered below are to be interpreted as Stieltjes integrals. However, since $f(x,y)$ depends on a neighborhood of a point it is likely that $f(x,y)$ is fairly smooth.

The distribution of the estimates of yield for the whole plot will then be

$$(2.1) \qquad \frac{1}{\sigma ab\sqrt{2\pi}} \int_0^a \int_0^b e^{-\frac{1}{2}([\xi - f(x,y)]^2/\sigma^2)}\, dydx$$

which, if $f(x,y)$ satisfies certain conditions as to continuity can be written as

$$(2.2) \qquad \frac{e^{-\frac{1}{2}(\xi^2/\sigma^2)}}{\sigma\sqrt{2\pi}} (A_0+A_1\xi+A_2\xi^2+A_3\xi^3+\cdots)$$

where

$$(2.3) \quad A_n=\frac{1}{ab}\int_0^a\int_0^b \frac{[f(x,y)]^n}{n!} e^{-\frac{1}{2}([f(x,y)]^2/\sigma^2)}\, dydx.$$

Distribution (2.2) must be carried to at least the ξ^3 terms in order to adequately represent the observed distributions of yields given in the first part of this section. It appears that (2.1) must be regarded as the fundamental distribution of random errors for a yield trial plot. In certain special cases, (2.2) may suffice. When $f(x,y)=k$, k a constant, (2.2) will reduce to a normal curve. In view of the above data on the heterogeneity of apparently very homogeneous soils it seems that only very rarely will $f(x,y)$ even approach a constant value and in some cases may be badly discontinuous.

If it is assumed that the first four terms of (2.2) adequately represent the distribution of the random errors and the method of moments is used in fitting then the constants of (2.2) have the following values in terms of moments of the distribution.

$$(2.4) \qquad \sigma^2=\sqrt{\mu 4/3}$$

$$A_0=1+\frac{\sqrt{\mu_4/3}-\mu_2}{2\sqrt{\mu_4/3}}$$

$$A_1=-\frac{\mu_3}{2(\mu_4/3)}$$

$$A_2=\frac{-3(\sqrt{\mu_4/3}-\mu_2)}{2\mu_4}$$

$$A_3=\frac{\mu_3}{6(\mu_4/3)^{3/2}}$$

A curve of the type (2.2) has been fitted to the data of Table II. The grouped frequency distribution and the theoretical normal curve and the curve of type (2.2) are shown in Chart 1. The curve of type (2.2) fits certain features of the grouped distribution much better than does the normal curve. The same is true of the data of Table I.

3. *Distribution of numbers attacked in field trials.* Salmon[3] has shown that the numbers of heads of wheat attacked by wheat bunt, for instance, in field trials is more variable than is indicated by assuming a binomial distribution. A possible explanation of this fact is that the probability of attack varies from point to point in the plots as discussed by Neyman[11] in connection with larval infestations and as indicated by Table IV and that the probability of attack may well be a function of time. There is probably a more or less definite period of susceptibility. At the beginning and end of the period the probability of attack may be small and become larger in the central part of the period. The simplest such distribution of probabilities is parabolic. In other cases, it is conceivable that the probability of attack remains constant throughout the susceptible period.

When some of the plants have been attacked these plants must be deducted from the number of the plants that are susceptible to attack. If n denotes the number of plants exposed to attack then n is also a function of time.

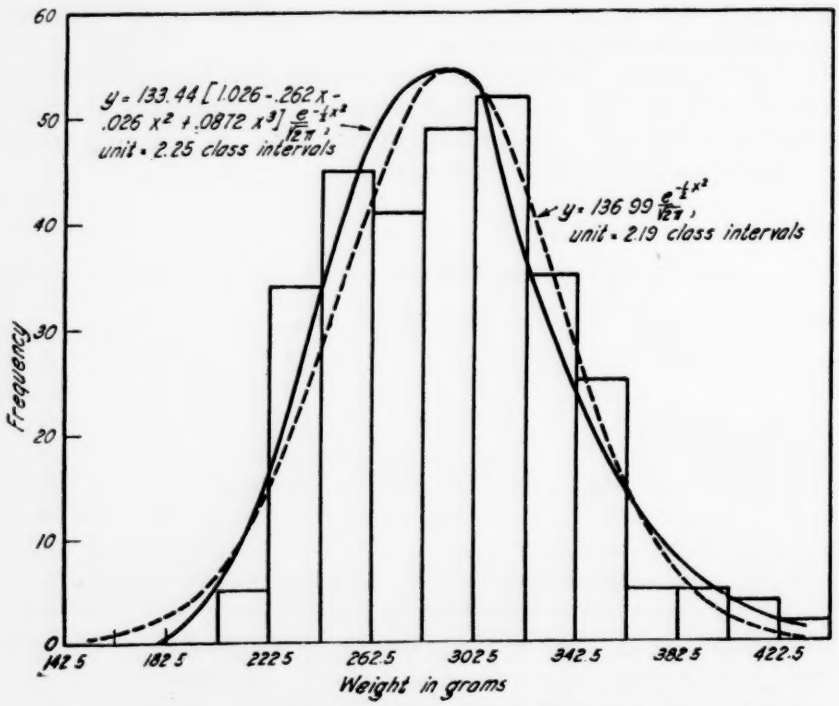

CHART 1. Comparison of the grouped frequency distribution of the yields of wheat on 300 75-square foot plots with a normal distribution and with the distribution of type (2.2).

To see the probable effect of these considerations assume that n is large, p is not too small and consider a small interval of time, Δt. Then the probability of the number, x, being attacked in the first interval will be approximately

$$(3.1) \qquad \frac{1}{\Delta t\sqrt{np_1q_1}\sqrt{2\pi}}\, e^{-\frac{1}{2}((x-np_1)^2/np_1q_1)}\,\Delta t$$

Thus the total probability of a number, x, being attacked is

$$(3.2) \quad \lim_{\substack{m\to\infty \\ \Delta t\to 0}} \sum_{i=1}^{m} \frac{1}{m\Delta t}\,\frac{1}{\sqrt{np_iq_i}\sqrt{2\pi}}\, e^{-\frac{1}{2}((x-np_i)^2/np_iq_i)}\,\Delta t$$

$$= \frac{1}{t_m-t_0}\int_{t_0}^{t_m} \frac{e^{-\frac{1}{2}([x-np(t)]^2/np(t)[1-p(t)])}\,dt}{\sqrt{np(t)[1-p(t)]}\sqrt{2\pi}}$$

where $p(t)$ is written to indicate that p depends on time. The exact form that (3.2) takes depends of course on the form of $p(t)$. If $p(t)$ is a constant (3.2) is normal, but if $p(t)$ is not constant then (3.2) is spread out as compared with a normal distribution. This spreading effect is well known under the name of the Lexis distribution.[10]

If the above restrictions are removed the probability of a number x being attacked in the interval Δt_i is

$$(3.3) \qquad \frac{1}{\Delta t_i}\begin{pmatrix} n(t_i) \\ x \end{pmatrix}[p(t_i)]^{n(t_i)-x}[q(t_i)]^x\Delta t_i$$

and the total probability of a number, x, being attacked in the time t_0 to t_m is.

$$(3.4) \qquad \frac{1}{t_m-t_0}\int_{t_0}^{t_m}\begin{pmatrix} n(t) \\ x \end{pmatrix}[p(t)]^{n(t)-x}[1-p(t)]^x dt$$

or if p is very small (3.3) and (3.4) should be replaced by the expressions for the Poisson distribution. If it is considered that p depends on ξ and η or position as well as on time, then (3.4) becomes a triple integral with respect to t, ξ, and η.

It follows that only rarely will the distribution of the numbers attacked in field trials be distributed as the Poisson or binomial laws, but will usually be distributed according to laws that have greater spreads or variabilities.

4. *Implications of the increased variability of the fundamental distributions.* The general effect of admitting the demonstrated more variable fundamental distributions is to make large observed differ-

ences more probable than is indicated by the usual analysis of variance. The assumption of a normal distribution for the random errors tends to overestimate significance as does the assumption of no correlation between the fertility levels of the various plots. Both effects operating at once may cause a very serious over-estimation of significance.

Certain properties of a soil can be accurately determined at a "point". In general there is a pronounced correlation between the factors influencing plant growth at the same "point." It is possible that with a sufficiently thorough inspection that plots of equal "soil fertility" could be selected by means of only a few criteria. Even though this could be done the distribution of random errors would still not be normal but be of the form (2.1)

It is desirable to get as good a notion as possible of the effect of assuming (2.1) instead of a normal distribution on the usual analysis of variance test. If (2.1) can be written as (2.2) then something can be said. It is known that if a random variable is distributed as (2.2) that the totals of samples of n drawn from (2.2) are also so distributed.[12] It should be noted that (2.2) is essentially a Gram-Charlier series or Edgeworth's generalized error distribution. Thus (2.2) applies to yields from any sized plot or any combination of plots. The distribution of variances of samples from (2.2) is not known except for samples of 2 and 3.[13] However, the distribution of the second moments of samples of n measured from a fixed point is known.[13] Some idea of the spreading effect of (2.2) can be obtained by comparing the distributions of $z=u_1/u_2$ under the two assumptions where u_1 and u_2 are independent estimates of the second moment based on n_1 and n_2 observations respectively. It should be noted that the coefficients for (2.2) will differ for n_1 and n_2.

If a normal fundamental distribution is assumed then z is distributed as

$$(4.1)\qquad \frac{4n_1^{n_1/2}n_2^{n_2/2}\,\Gamma\left(\frac{n_1+n_2}{2}\right)z^{(n_1-2)/2}}{\Gamma\left(\frac{n_1-2}{2}\right)\Gamma\left(\frac{n_2-2}{2}\right)(n_1z+n_2)^{(n_1+n_2)/2}}$$

while if (2.2) with three terms is assumed to be the fundamental distribution z is distributed as

$$\sum_{s=0}^{n_1}\sum_{r=0}^{n_2}\left[\binom{n_1}{s}\binom{n_2}{r}B_0^{s}A_0^{r}B_2^{n_2-s}A_2^{n_2-r}n_1^{3n_1-2s+2}n_2^{3n_2-2r+2}\right.$$

$$(4.2)\quad \sum_{s=0}^{n_1}\sum_{r=0}^{n_2}\left[\frac{\Gamma\left(\frac{3n_1+3n_2-2s-2r+4}{2}\right)z^{(3n_1-2s)/2}}{\Gamma\left(\frac{3n_1-2s}{2}\right)\Gamma\left(\frac{3n_2-2r}{2}\right)(n_1z+n_2)^{(3n_1+3n_2-2s-2r+4)/2}}\right]$$

The ratio of the estimates of the second moments are being used instead of estimates of the variances as is done in the analysis of variance. Expression (4.2) becomes (4.1) under certain conditions but in general is spread out wider.

Extended discussion of formula (4.2):

Consider two populations of type (2.2) consisting of the first three terms

$$(A)\qquad \frac{e^{-\frac{1}{2}\xi^2}}{\sqrt{2\pi}}\quad (A_0+A_1\xi+A_2\xi^2)$$

$$(B)\qquad \frac{e^{-\frac{1}{2}\xi^2}}{\sqrt{2\pi}}\quad (B_0+B_1\xi+B_2\xi^2).$$

The populations (A) and (B) are considered because the coefficients of the population of type (2.2) will vary with the size of the plots considered in the different parts of an analysis of variance. To be strictly comparable to (4.1) the B's should be written as A's in (4.2).

The distribution of the second moments, u_1, of samples of n_1 from population (B) are distributed as proportional to

$$(C)\qquad \sum_{s=0}^{n_1}\binom{n_1}{s}\frac{B_0{}^sB_2{}^{n_1-s}n_1e^{-n_1u_1}(n_1u_1)^{(3n_1-2s)/2}}{\Gamma\left(\frac{3n_1-2s}{2}\right)}$$

and the distribution of the second moments, u_2, of samples of n_2 from population (A) are distributed as proportional to

$$(D)\qquad \sum_{r=0}^{n_2}\binom{n_2}{r}\frac{A_0{}^rA_2{}^{n_2-r}n_2e^{-n_2u_2}(n_2u_2)^{(3n_2-2r)/2}}{\Gamma\left(\frac{3n_2-2r}{2}\right)}.$$

The results (C) and (D) are simply Theorem II of (13) stated in a better form. The estimates u_1 and u_2 are assumed independent so that their joint frequency distribution is the product of (C) and (D). Make the transformation $u_1=u_2z$, $u_2=u_2$, $du_1=u_2dz$ and then integrate out with respect to u_2. The formula (4.2) results.

5. *Summary.* It has been shown that the fundamental distributions for field trial data are usually spread more widely or are more variable than is assumed in the classic tests of significance which are usually applied to data of these types. The effect of this spreading is to make the classic tests over-estimate the significance of observed differences. Expressions are given that permit an indication of this over-estimating tendency to be given under specified sets of conditions.

The spreading of the fundamental distributions is due to the fact that the fertility level or probability of attack varies from point to point in each plot and that these quantities may also vary with time. Proebsting[14] and [15] has shown that the soil solution changes markedly with time.

It is suggested that it would be very desirable to make detailed examinations of plots and correlate these observations with yields or numbers attacked recorded for very small units of area and for short intervals of time for land that is commonly used for agricultural trials. It is only by this means that more exact information may be obtained relative to the differences that may be considered significant.

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After they have said that the study of Latin in school is a waste of time the majority of people rest their case. In an equally off-hand way they dismiss the importance of mathematics, except for those who intend to go into engineering or similar careers.

Such people completely miss the point.

For most youngsters the mastery of Latin and solid geometry is no more "useful" than skill in horseback-riding or contract bridge. Much depends on what we mean by useful.

The kind of mind that can master Latin and higher mathematics is a trained and disciplined mind that can be concentrated on new and different problems and can probably solve them. Such a mind can attack contract bridge, for example, and comprehend the fine points of the game more readily than a mind that has never been compelled to function with precision.

Aside from its social aspects, horseback-riding is an excellent training in muscular coordination. A boy might spend time and money in learning to ride. His time would not be wasted even though he never rode in his adult life. In whatever physical activity he engaged, his training in horsemanship would be valuable.

Just so, after studying Latin and mathematics, a young man's mind is better able to cope with any mental problem that may be presented. If he has learned to solve original problems in geometry, he will know something about how to attack any other original problem, whether in business, law or science.—*The Imperial Magazine*, September, 1941.

The Trisection Problem

By Robert C. Yates
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CHAPTER IV

APPROXIMATIONS

Sufficient for any purpose are a large number of constructions by straightedge and compasses which, although simple, give remarkably good approximations [**49**] for trisecting a given angle. Many of these are of long standing and exhibit considerable ingenuity of construction. Individuals who think they have found exact methods of trisection of the general angle by straightedge and compasses have actually found nothing more than approximations. Of course, a large number of these attempts yield very accurate results and to the eye the drawings appear successful indeed. Dependence on such a physical impression, however, often brings about unfortunate conclusions.

1. An Unending Construction

Fialkowski [**19**] in 1860, based an approximation upon the series:

$$S_n = 1/2 - 1/2^2 + 1/2^3 - 1/2^4 + \cdots \pm 1/2^n. \tag{1.1}$$

If this series is multiplied through by (1/2), it becomes:

$$S_n/2 = 1/2^2 - 1/2^3 + 1/2^4 - 1/2^5 + \cdots \pm 1/2^{n+1}.$$

Now, by adding the two equations, we have the compact expression for the sum:

$$3S_n/2 = 1/2 \pm 1/2^{n+1},$$

or

$$S_n = (1/3)(1 \pm 1/2^n).$$

As n is allowed to grow larger, it is evident that the value of S_n grows nearer equal to 1/3. Thus, for example, on taking eight terms of the series, $n=8$, S_n differs from 1/3 by an amount equal to $1/2^8$; taking $n=9$, S_n differs from 1/3 by $1/2^9$; etc.

For the application of this to the approximate trisection of $AOB = \Phi$, let S_n represent the ratio Θ_n/Φ, where Θ_n is the angle to be constructed at the nth step. That is,

$$\Theta_n = (1/3)(1 \pm 1/2^n) \cdot \Phi. \tag{1.2}$$

This is the fourth in a series of five chapters. The last will appear in an early issue.

It is apparent from this that the larger we take n, the nearer will θ_n equal $\Phi/3$, or $AOB/3$. Let us agree that a positive angle is to be measured from OB towards OA, a negative angle in the opposite

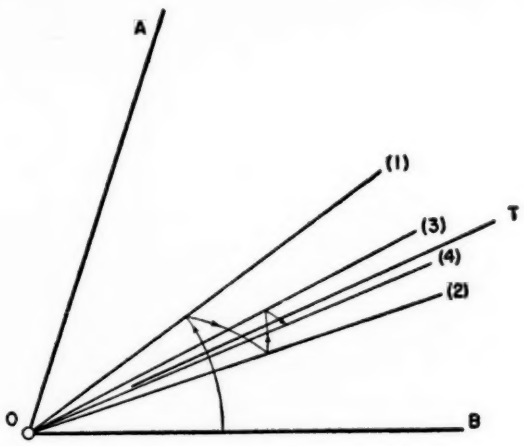

FIG. 32

direction. The steps in the construction are indicated in (1.2): first, bisect angle AOB, obtaining the line (1); then bisect angle $BO(1)$, obtaining the line (2); etc., alternately adding and subtracting as shown. The process leads step by step toward the actual position of the trisecting line OT. It must be realized, however, that no matter how large n be chosen, θ_n will differ by *some* amount from $\Phi/3$. For example, if eight steps be taken for the angle $AOB = \Phi = 60°$, equation (1.2) gives:

$$\theta_8 = (60°/3)(1 - 2^{-8}) = 20° - 20°/2^8,$$

so that θ_8 is too small by less than five minutes.

2. *Approximation of von Cusa and Snellius*

Consider the sector AOB of the unit circle with the segments AC and BD drawn perpendicular to OB. If the central angle be measured in radians, then

$$AC = \sin\theta, \quad \text{arc } AB = \theta, \quad BD = \tan\theta.$$

Evidently these lengths satisfy the inequality:

(2.1) $$\sin\theta \leq \theta \leq \tan\theta.$$

Nikolaus von Cusa, who lived in the 15th century, noticed that the quantity:

(2.2) $$(1+n)\sin\theta/(n+\cos\theta)$$

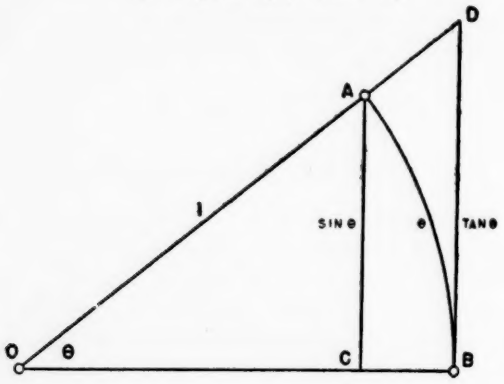

FIG. 33

was, for $n=0$, equal to $\tan\theta$ and thus, by (2.1), *greater* than θ; for n tending to infinity the ratio becomes equal to $\sin\theta$ and thus *less* than θ. He accordingly proposed the following question: For what value of n is this quantity the best approximation for θ? The answer he gave was $n=2$. Let us see for ourselves.

If the quantity (2.2) is to be a good approximation for θ then we should be able to make the difference between the quantity and the exact value of θ as small as we please. This difference is:

$$D=[(1+n)\sin\theta/(n+\cos\theta)]-\theta$$
$$=[(1+n)\sin\theta-n\theta-\theta\cos\theta]/(n+\cos\theta).$$

By using well known expressions for $\sin\theta$ and $\cos\theta$ directly in terms of θ this difference can be written in the form:

$$D=\frac{[(1+n)(\theta-\theta^3/3!+\theta^5/5!-\cdots)-n\theta-\theta(1-\theta^2/2!+\theta^4/4!-\cdots)]}{(n+\cos\theta)}$$

or $$D=(2-n)\theta^3/6(n+\cos\theta)+(n-4)\theta^5/120(n+\cos\theta)+(\ )\theta^7+\cdots.$$

If θ, measured in radians, is numerically less than 1, the terms in the right member diminish rapidly as the series is extended since a power of a positive number less than 1 decreases as the power increases.

It may be seen that the result, $n=2$, arrived at empirically by von Cusa, is in fact the best possible choice for $|\theta|<1$. For, by

putting $n=2$ in this last equation, the first power of θ to appear is the 5th and the whole right hand member is considerably less than if the 3rd power were present. Willebrod Snellius, in the next century, was the first to consider the von Cusa question on a rigorous basis and equal credit for the method should belong to him.*

We may thus establish the approximation formula:

$$\theta = 3 \sin \theta/(2+\cos \theta).$$

In order to see the geometrical meaning contained in the formula,

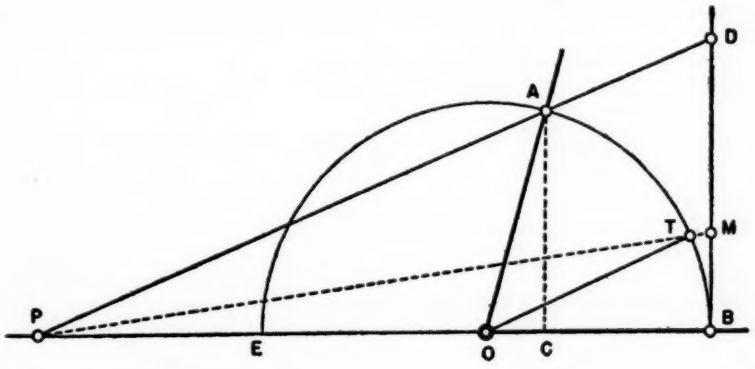

FIG. 34

extend the diameter BE of the unit circle, Fig. 34, the distance 1 to P. The line PA strikes the tangent at D. From the similar triangles PCA and PBD, we have the proportion:

$$AC/PC = BD/PB,$$

or, since $AC = \sin \theta$, $PC = 2+\cos \theta$, $PB = 3$:

$$BD = 3 \sin \theta/(2+\cos \theta).$$

Thus the approximation made by von Cusa was in replacing the arc length subtending angle AOB by the tangent length BD.

For the approximate trisection of angle AOB we need only locate the point M so that $BM = BD/3$ and join it to P. The line PM will cut the unit circle in T. The line joining T to O thus approximately trisects angle AOB. A table of errors for angles from 0° to 90° follows:

*Snellius in his *Cyclometria* figured π by using polygons up to 5,242,880 sides.

Angle	*Error*	*Angle*	*Error*	*Angle*	*Error*
10°	13″	40°	15′ 4″	70°	1°28′
20°	1′51″	50°	30′ 6″	80°	2°17′
30°	6′20″	60°	53′20″	90°	3°27′

3. *Dürer's Approximation*

Extremely elegant in both its simplicity and accuracy is the early approximation of Albrech Dürer which appeared [**15**] in 1525. Let the chord and the subtended arc of the unit circle be constructed upon the given angle $AOB = 2\theta$. The points M_1, M_2 are located dividing

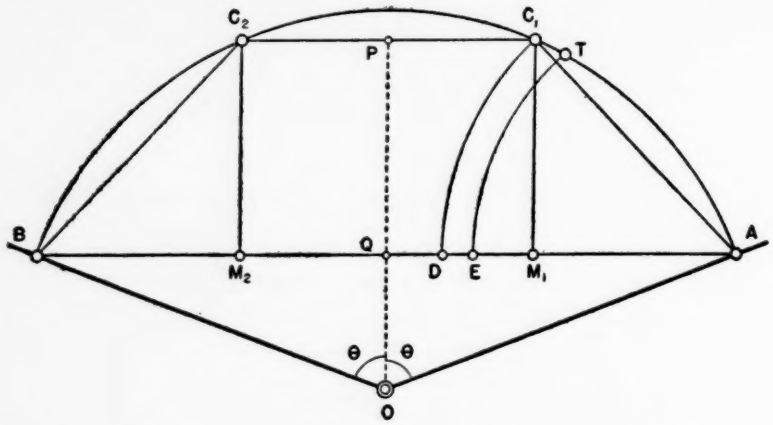

FIG. 35

the chord into three equal parts and at these points perpendiculars are erected to meet the arc in C_1 and C_2. It was Dürer's plan to construct the average between the chord lengths, AC_1, C_1C_2, and C_2B in as few steps as possible. To this end, he described the arc C_1D with A as center and AC_1 as radius. Then, locating the point E such that $EM_1 = 2(DE)$, a second arc, center at A and radius AE, was drawn intersecting the circle at T. The line OT then was his approximate trisecting line.

In order to show the accuracy of this construction, a little trouble lies in store, but the bother is compensated by the character of the method. Draw the bisecting line OP so that PC_1 is parallel to the chord AB. The following relations are determined with the help of the figure:

$$OQ = \cos\theta; \quad AQ = \sin\theta; \quad AB = 2\sin\theta; \quad AM_1 = 2\sin\theta/3;$$

$$PC_1 = AQ - AM_1 = \sin\theta/3; \quad OP = \sqrt{(OC_1^2 - PC_1^2)} = (1/3)\sqrt{(9-\sin^2\theta)};$$

$$PQ = C_1M_1 = OP - OQ = (1/3)\sqrt{(9-\sin^2\theta)} - \cos\theta;$$

$$AC_1 = AD = \sqrt{(AM_1^2 + C_1M_1^2)} = \sqrt{(2/3)}\sqrt{2+\cos^2\theta - \cos\theta\sqrt{(8+\cos^2\theta)}}.$$

Now by construction:

$$AT = AE = AM_1 + M_1E = AM_1 + (2/3)(M_1D)$$
$$= AM_1 + (2/3)(AD - AM_1) = (1/3)(AM_1 + 2\cdot AD),$$

which is precisely the average of the chords AC_1, C_1C_2, C_2B, which Dürer planned. We have for this, on substituting the expressions for AM_1 and AD in terms of θ:

$$AT = (2/9)\sin\theta + (2\sqrt{2}/3\sqrt{3})\cdot\sqrt{2+\cos^2\theta - \cos\theta\cdot\sqrt{8+\cos^2\theta}},$$

the length of the approximate trisecting chord in terms of the given angle. Thus, since

$$2\sin(AOT/2) = AT \quad \text{or} \quad AOT = 2\cdot\arcsin(AT/2),$$

we have:

$$\angle AOT = 2\cdot\arcsin\left[(1/9)\sin\theta + \sqrt{(2/27)}\cdot\sqrt{2+\cos^2\theta - \cos\theta\cdot\sqrt{8+\cos^2\theta}}\right]$$

as the approximate third part of the given angle. A table of errors for angles between 0° and 180° gives an indication of the remarkable accuracy:

Angle	*Error*	*Angle*	*Error*
60°	1″	140°	5′37″
90°	18″	150°	9′ 4″
120°	1′56″	180°	31′38″

Before passing on, we must recall to the reader's mind a few of the accomplishments of this remarkable man. He is perhaps best known for his etchings and paintings. A champion of the art of perspective, he organized the information developed up to his time into the first text book on the subject. His etching, *Melancholia*, is to be found in many present day books on architecture and mathematics.

4. *Karajordanoff's Approximation*

Dropping down several centuries to recent times, we find a simple approximation discovered by Karajordanoff in 1928 [**5**]. The circles of radii 1 and 2 are drawn about the angle AOB. The tangent to the

unit circle at A meets the chord BC in D, where C is the midpoint of AB. The line through D parallel to OB intersects the larger circle in T and OT is the approximate trisecting line.

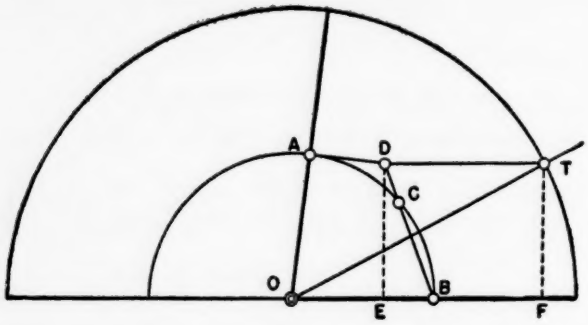

FIG. 36

We seek the error committed in trisecting various angles by this method. Making use of rectangular coordinates with O for origin and OB as $X-$axis, the coordinates of the several points are:

$A : (\cos 2\theta, \sin 2\theta)$; $B : (1,0)$; $C : (\cos \theta, \sin \theta)$.

The line through B and C has for its equation:

$$x = 1 - y \cdot \tan(\theta/2)$$

while the tangent to the smaller circle at A is (with negative reciprocal slope of OA):

$$x \cos 2\theta + y \sin 2\theta = 1.$$

The two lines meet at D, the $y-$coordinate, DE, of which is obtained by eliminating x between the two:

$$y = (1 - \cos 2\theta)/(\sin 2\theta - \cos 2\theta \cdot \tan \theta/2).$$

This, by construction, is the length of the segment FT and, since $OT = 2$:

$$\angle FOT = \arcsin[(1 - \cos 2\theta)\sin \theta/2(\cos \theta - \cos 2\theta)].$$

Errors for angles between 0° and 90° follow:

Angle	*Error*	*Angle*	*Error*	*Angle*	*Error*
10°	1″	40°	45″	70°	2′12″
20°	6″	50°	1′19″	80°	1′50″
30°	21″	60°	1′53″	90°	0

Although this method of Karajordanoff is not as accurate as Dürer's for some angles, it has a redeeming feature: the error does not always increase as the given angles increase to 90°. A maximum error occurs at 70°15′ with a value less than two and one-third minutes.

5. *Kopf-Perron Approximation*

A comparative study of the arc lengths of a particular circle and the trisecting Limaçon of Pascal led Kopf in 1919 to the following method, which was refined somewhat later by Perron [**40**] and d'Ocagne [**38**].

The unit circle cuts the sides of the given angle at A and B. The midpoint D of OC is marked, PC is taken equal to 1, and DF as one-third the length DE; that is, since $OD=1/2$, $OE=1$:

$$ED=\sqrt{3}/2 \quad \text{and} \quad DF=\sqrt{3}/6.$$

With center at F: $\left(-\dfrac{3+\sqrt{3}}{6}, 0\right)$

and radius $FB=1+(3+\sqrt{3})/6$, the arc BA' is constructed. The line CA meets this arc in A' and angle $A'PB$ is approximately $\angle AOB/3$. The errors committed for various angles have already been tabulated by Perron:

Angle	*Error*	*Angle*	*Error*	*Angle*	*Error*
12°	0.18″	48°	8.58″	84°	8.47″
24°	1.38″	60°	13.08″	90°	0
36°	4.23″	72°	14.76″		

The error, as in the method of Karajordanoff, is not an increasing one. The maximum of 14.9″ occurs at 69°57′40″.

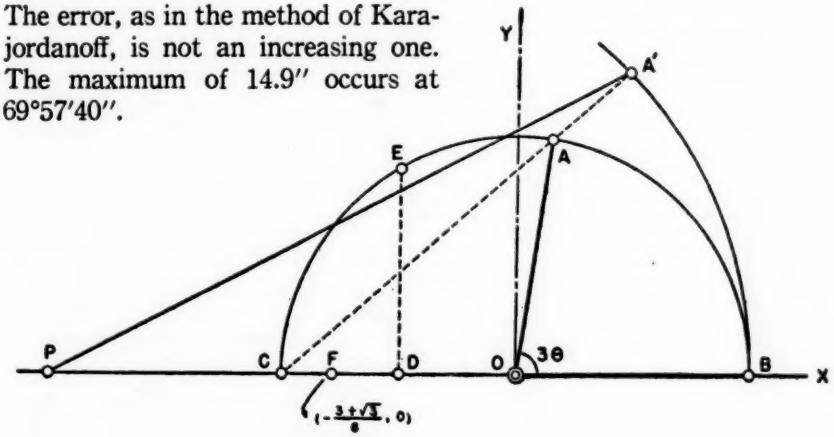

FIG. 37

6. *Approximation of D'Ocagne*

A recent and extremely simple method that is surprisingly accurate for small angles is given by d'Ocagne [38]. From the midpoint C of the radius of the unit circle, the line CM is drawn to the midpoint of the arc of the given angle AOB. Then angle MCB is approximately one-third angle AOB.

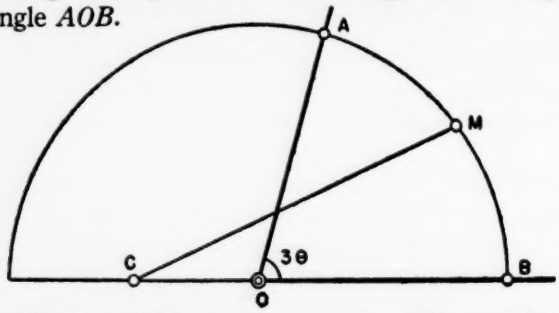

FIG. 38

The coordinates of M are $(\cos\theta, \sin\theta)$. Thus we find:

$$<MCB = \text{arc tan}\,[2\sin\theta/(1+2\cos\theta)] = \text{arc cot}[\cot\theta + \cot(\theta/2)].$$

The errors are:

Angle	*Error*	*Angle*	*Error*	*Angle*	*Error*
10°	1.5″	40°	1′48″	70°	10′
20°	2.64″	50°	3′26″	80°	15′
30°	45″	60°	6′14″	90°	21′41″

7. *Comparison of Methods*

The following chart affords a comparison of the methods of approximation given in this chapter. Errors are plotted against corresponding angles.

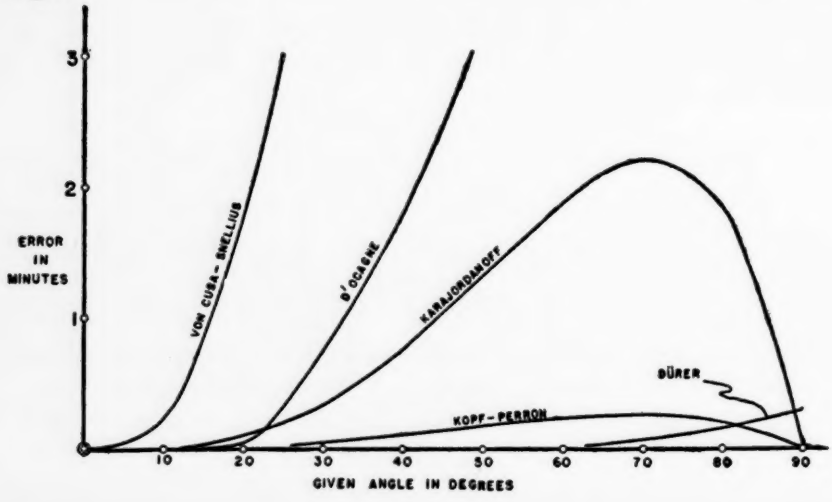

FIG. 39

Two Ways of Setting Up the Symmetric Correspondence $C_{2,2}$ and a Relation Connecting Them

By Sister Leonarda Burke
Regis College, Weston, Massachusetts

Sturm[1] has collected the results of the most important investigations on the theory of correspondence. The earliest of these investigations was published in 1864 as an explanation of the general method of approach offered by Chasles.[2] Other publications by Cremona,[3] Clebsch,[4] Weyr,[5] and Zeuthen[6] appeared from time to time during the fifty years following. Godeau[7] mentions a special $C_{2,2}$ between lines of two bundles. A dissertation prepared by Sister Catharine Francis Galvin[8] deals with correspondence between points on the same curve. Three correspondences are treated in this latter paper, a $C_{3,3}$ and two cases of the $C_{4,4}$. The writer treats the same problem for $n=3$ as is here treated for $n=2$.

The present paper deals with the symmetric correspondence $C_{2,2}$. Such a correspondence may be set up on a conic K_1 between two parameters the tangents at which intersect on a second conic K_2, thus providing the general $C_{2,2}$. A like correspondence may be set up on a rational bicuspidal quartic between parameters at residual intersections on a tangent. The correspondence between these parameters is to be regarded as a special type. The $C_{2,2}$ set up by the first method

[1] *Die Lehre von den Geometrischen Verwandtschaften* (Leipzig, Teubner, 1908), Vol. I, pp. 225-319.

[2] "Considérations sur la méthode générale exposée dans la séance du 15 février," *Comptes Rendus*, Vol. 58 (1864), p. 1175.

[3] "Un teorema intorno alle forme quadratiche non omogenee fra due variabile," *Rendiconti dell' Istituto Lombardo*, Ser. I, Vol. IV (1867), p. 199.

[4] "Uber die Singularitäten algebraischer Curven," *Journal für die reine und angewandte Mathematik*, Vol. 64 (1865), p. 98.

[5] "Bestimmung der Anzahl involutorischer Elementenpaare einförmiger mehrdeutiger Gebilde," *Journal f. Math.*, Vol. 74 (1872), p. 189.

[6] "Nouvelle démonstration de théorèmes sur des séries de points correspondants sur deux courbes," *Mathematische Annalen*, Vol. 3 (1871), p. 150.

[7] "*ABC* étant une triangle donné, soit ω parabole qui touche les côtés *AB*, *AC*, et donc le foyer est une point quelconque du côté *BC*. Trouver: le lieu géometrique du point du rencontre des normales à ω en *B'* et *C'*." *Mathesis*, Vol. 41 (1927), p. 44.

[8] Dissertation: *Two Geometrical Representations of the Symmetric Correspondence, $C_{n,n}$, with their Interrelations*, The Catholic University of America, 1938.

reduces to the second type if the two conics satisfy one invariant condition which it is the object of the present paper to derive.[9]

The system consisting of the two conics can satisfy ten conditions. Eight of these conditions are given by the general ternary collineation. The rational quartic can satisfy eleven conditions in general, but if it is required to have two cusps, only nine additional conditions can be imposed on it. Thus the system of two conics has two absolute invariants while the rational quartic has but one. It is evident that the system consisting of the two conics will have to satisfy one invariant condition in order that the two correspondence equations obtained will be equivalent.

We take for the conic K_1 the norm-conic N_2

$$4x_0x_2=x_1^2$$

which in parametric form becomes

$$\rho x_0=1$$
$$\rho x_1=2\lambda$$
$$\rho x_2=\lambda^2.$$

We take for the conic K_2

$$b_{11}x_1^2+2b_{12}x_1x_2+2b_{20}x_2x_0+2b_{01}x_0x_1=0.$$

Thus we take two of the four intersections of the two conics as vertices of the reference triangle, and choose the values of the parameters there to be $\lambda=0$, $\lambda=\infty$.

There are four values of λ each of which has a distinguishing property, namely, the property that one correspondent coincides with it. The penultimate case discloses that these values of λ are at the four intersections of K_1 and K_2. If $\lambda_1^{(2)}$ approaches P in Figure I, $\lambda_2^{(1)}$ approaches P. The four points of the bicuspidal rational quartic which have this property are the two cusps and the two points of tangency of the (unique) double tangent. If λ_1 approaches Q in Figure II, $\lambda_2^{(1)}$ will approach Q.

Hence we take the parametric representation of the quartic so that the cusps shall be at $\lambda=0$, $\lambda=\infty$. Further we place the cusps at the two vertices of the reference triangle and take the tangents thereat as sides of this triangle. The parametric equations become

$$\rho' x_0=\lambda^3(\lambda-\alpha)$$
$$\rho' x_1=\lambda-1/\alpha$$
$$\rho' x_2=\lambda^2.$$

[9] Dr. Aubrey E. Landry, Chairman of the Mathematics Department of the Catholic University of America, offered many helpful suggestions during the development of the problem.

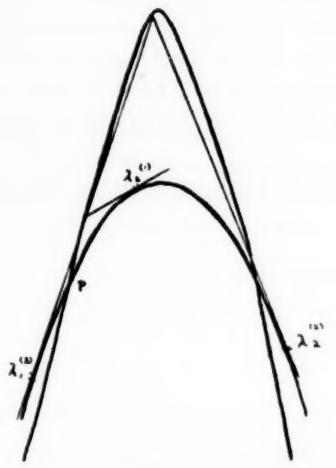

Figure I

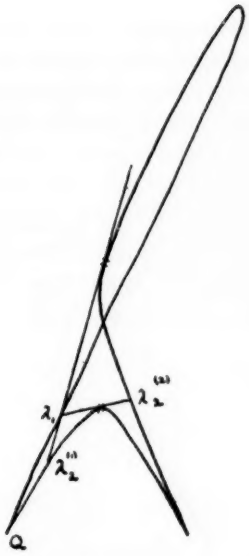

Figure II

The coordinates are exhibited as functions of a single quantity α. Thus there remains one arbitrary constant which is all this curve possesses.

The correspondence equations are now comparable.

If tangents are drawn at λ_1 and λ_2 to the conic $K_1(=N_2)$, these tangents will intersect in a point whose coordinates are (s_0, s_1, s_2), where $s_0=1$, $s_1=\lambda_1+\lambda_2$, $s_2=\lambda_1\lambda_2$. Therefore, the correspondence equation[10] between λ_1 and λ_2 which expresses that the tangents at λ_1 and λ_2 intersect on K_2 is obtainable from the equation of K_2 by replacing x_i by s_i.

$$b_{11}s_1^2+2b_{12}s_1s_2+2b_{20}s_2s_0+2b_{01}s_0s_1=0.$$

The class of the rational quartic is lowered by two by the existence of two cusps; correspondingly, the $C_{4,4}$ between parameters on a tangent to a general R^4 reduces to a $C_{2,2}$ by the factoring out of a pair of linear factors arising from the pencils of lines through the cusps. We obtain

$$(\sigma_1-\alpha\sigma_2)(\sigma_1-\alpha)[\sigma_1^2-\sigma_1(\alpha\sigma_2+\alpha)-\sigma_2(4-\alpha^2)]=0.$$

Rejecting the first two factors for the reason stated, the remaining equation is the desired correspondence equation, the $C_{2,2}$ set up by the $R^{4\cdot 2^k}$.

$$\sigma_1^2-\sigma_1(\alpha\sigma_2+\alpha)-\sigma_2(4-\alpha^2)=0.$$

[10] Meyer, W. F.: *Apolarität und Rationale Curven*, (Tübingen, Fues, 1883), p. 45.

We replace σ_i by x_i after making the equation homogeneous by the insertion of σ_0, where $\sigma_0 = 1$. The equation which results is the equation of a conic which with N_2 sets up the special case of the $C_{2,2}$. The general case of the N_2 with any conic, say with K_2 in the early part of this paper, shows no equivalence since the fundamental invariants of this conic pair are mutually independent. There is one relation connecting the four fundamental invariants in the special case. We proceed to discover that relation.

The four invariants of N_2 and the conic formed as explained from the $C_{2,2}$ set up on the R^{4+2k},

$$N_2 : 4x_0x_2 = x_1^2$$

$$K_2' : x_1^2 - \alpha x_1x_2 - \alpha x_1x_0 - x_2x_0(4-\alpha^2) = 0$$

are

$$\Delta = -4$$

$$\Theta = 2\alpha^2 - 12$$

$$\Theta' = \frac{-\alpha^4 + 12\alpha^2 - 48}{4}$$

$$\Delta' = \alpha^2 - 4.$$

We form the three quantities Δ'/Θ', Θ'/Θ, Θ/Δ. If a homogeneous relation can be found in these three quantities, it will, when cleared of fractions, be homogeneous in both sets of coefficients, and therefore an (rational integral) invariant relation between the two conics. For this purpose, we write

$$\alpha^2 = \rho, \quad \Delta'/\Theta' = x, \quad \Theta'/\Theta = y, \quad \Theta/\Delta = z.$$

Thus

$$x = \frac{4(\rho - 4)}{-\rho^2 + 12\rho - 48}$$

$$y = \frac{-\rho^2 + 12\rho - 48}{8\rho - 48}$$

$$z = \frac{2\rho - 12}{-4}.$$

These expressions, cleared of fractions, become

$$kx = 32\rho^2 - 320\rho + 768$$

$$ky = \rho^4 - 24\rho^3 + 240\rho^2 - 1152\rho + 2304$$

$$kz = 4\rho^4 - 96\rho^3 + 912\rho^2 - 4032\rho + 6912.$$

We interpret x, y, z as homogeneous coordinates of a point in a plane and obtain the non-parametric equation of the quartic curve represented by following the method outlined by Richmond.[11] The relation thus obtained is the following irreducible quartic in x, y, z.

$$z^4-9yz^3+z^2(9xy+24y^2)+z(-16y^3-36xy^2)+27x^2y^2=0.$$

Thus the correspondence as we have set it up on the conics reduces to the correspondence set up between parameters at residual intersections of tangents to a rational quartic having two cusps. The relation stated may be expressed in terms of the four invariants:

Theorem: The condition that the correspondence $C_{2,2}$ set up between lines of a conic K_1 which intersect on a conic K_2 be of the special type set up between parameters on an $R_{4,2k}$ whose joining lines are tangent to the curve is

$$\Theta^6-9\Delta\Theta^4\Theta'+9\Delta^2\Theta^3\Delta'+24\Delta^2\Theta^2\Theta'^2-16\Delta^3\Theta'^3-36\Delta^3\Theta\Theta'\Delta'+27\Delta^4\Delta'^2=0.$$

[11] *Bulletin*, American Mathematical Society, November, 1916, p. 90.

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A Note on the Torsion Problem

By THOMAS JAMES HIGGINS
Tulane University

This note is written to point out in that standard work on elasticity, A. E. H. Love's *A Treatise on the Mathematical Theory of Elasticity*, a mis-statement which has stood since the second edition of this classic appeared in 1906. As by virtue of its inclusion in Love's text this mis-statement has subsequently appeared in several texts on applied mechanics and in numerous papers on Saint-Venant's torsion and flexure problems, the following is of historical and critical interest.

With reference to the shearing stress in a linear isotropic prism subjected to pure torsion, Love's statement (*op. cit.*, 2nd ed., p. 304; 4th ed., p. 316)

> "...; and therefore the greatest value of the shearing stress is found on the cylindrical boundary."

has appended to it the footnote

> "Boussinesq had supposed that the points of maximum shearing stress must be those points of the contour which are nearest the axis; but Filon later showed that this is not necessarily true."

Now this statement by Love evidently stems from the late Professor L. N. G. Filon's assertion (*Philosophical Transactions of the Royal Society*, Vol. 193A (1900), p. 332

> "The second point of importance which these tables bring out clearly is that the maximum strain and stress do not always occur, as most of the results obtained by DE SAINT-VENANT would lead one to suppose, and as THOMSON and TAIT (*Natural Philosophy*, Vol. 1, Part II, 710) and BOUSSINESQ (*Journal de Mathématiques*, Série II, Vol. 16, p. 200) assumed, at the point of the boundary nearest the center."
>
> "Indeed SAINT-VENANT himself, in his edition of NAVIER'S *Lecons de Mécanique* (§33, p. 313), has given an example to the contrary...."

Now note the actual statements of Thomson (*op. cit.*, §710) and Boussinesq (*loc. cit.*, p. 200).

> "M. de St. Venant also calls attention to a conclusion from his solutions which to many may be startling, that in his *simpler cases* the places of greatest distortion are those points of the boundary which are nearest to the axis of the twisted prism in each case, and the places of least distortion are those farthest from it."

> "...; ses valeurs maxima auront donc lieu *en général* sur les petits diamètres des sections, et ses valeurs minima sur les grands."

Now certainly, in view of the italicized words in these quotations (the writer's italics) neither of the latter can be interpreted (although Filon evidently did so) as a flat assertion that the maximum shearing stress *always* occurs at those points on the contour which are nearest the axis of torsion. Kelvin's own words speak to the contrary; Boussinesq's "en general" likewise: indeed, in all probability the latter's qualifying phrase stems from explicit knowledge of the case to the contrary mentioned above.

Some pages after the above-quoted paragraph Filon writing

> "M. BOUSSINESQ has given (*Journal de Mathématiques*, Series II, Vol. 16, p. 200) a sketch of a proof that the fail-points must be sought for 'sur les petits diamètres des sections.'"

takes issue with the validity of Boussinesq's so-called proof and points out where (in his estimation) the former erred. In reality, however, in his use of the words, proof and must, Filon himself is in variance with fact. Actually, as reference to the paper cited reveals, Boussinesq *indicated* in a well known fashion how one can account for the fact that the maximum stress can (and usually does) occur at points of the contour nearest the center of the section and *proved* the perfectly valid theorem quoted above from Love.

In light of the foregoing it is difficult to account for Filon's misinterpretations; it is equally obvious that Love's remark stemmed from reading Filon's statements and accepting the same without verification through reference to the sources mentioned. In this light, then, one would have here but an amusing instance of the pitfall that awaits every writer, no matter how expert in the field whereof he writes, were it not that the authority implicitly vested in Love's work has resulted in manifold publication of this falsity: for it is to be found in many of the important modern papers on the torsion and flexure problems.

And while one may facetiously remark that at least Boussinesq's name is thus kept green, he can scarcely deny that such ill-founded

perpetuation scarcely befits one of the distinguished applied mathematicians of his period, the author (in the paper cited) of the first truly comprehensive treatment of the torsion problem for prisms of multiply-connected section; nor disbelieve that Boussinesq himself would not, if possible, request that future writers desist in attributing to him that which he neither wrote nor held.

"Mathematics is an activity governed by the same rules imposed upon the symphonies of Beethoven, the paintings of Da Vinci, and the poetry of Homer. Just as scales, as the laws of perspective, as the rules of metre seem to lack fire, the formal rules of mathematics may appear to be without lustre. Yet ultimately, mathematics reaches pinnacles as high as those attained by the imagination in its most daring reconnoiters. And this conceals, perhaps, the ultimate paradox of science. For in their prosaic plodding both logic and mathematics often outstrip their advance guard and show that the world of pure reason is stranger than the world of pure fancy."—From *Mathematics and the Imagination* by Edward Kasner and James Newman. p. 362.

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The General Equation of the Second Degree

By K. P. WILLIAMS
Indiana University

1. This paper deals with the elementary theorem that, excepting special cases, any equation of the second degree in two variables represents a conic. The theorem is of much importance, but in texts on analytic geometry its proof is usually delayed until equations for the three conics in standard form have been obtained. The customary proof employs transformation of coordinates, which requires a number of preliminaries and introduces ideas that usually confuse the beginner. The treatment is different for central and non-central conics. Furthermore the procedure gives no simple direct solution of some questions that might be asked. For instance, if the eccentricity and one focus and directrix of an ellipse or hyperbola are known, it is quite reasonable to ask for the other focus and directrix. The solution of the problem by customary formulas leads to the calculation of several quantities not desired.

The present treatment attacks the problem directly without any preliminaries, for nothing is used except the general definition of a conic. The algebraic situation that arises has some interesting aspects.

2. The equation of the conic with the line $mx+ny+p=0$ for directrix, the point (α,β) for focus, and eccentricity e, is

$$(1) \qquad (x-\alpha)^2+(y-\beta)^2=e^2\frac{(mx+ny+p)^2}{m^2+n^2} .$$

The equation

$$(2) \qquad Ax^2+2Hxy+By^2+2Cx+2Dy+E=0$$

will represent a conic if it can be written in the form (1). The equation (2) may factor, and the condition for factorization, found by straight-

forward algebra, will be assumed known, and will appear later. But although (2) may factor, it may still be written in the form (1), as will appear later.

If equation (1) is expanded and coefficients are compared with those in (2) it is found that there are two distinct groups of equations, the first group being

$$\begin{aligned} m^2+n^2-e^2m^2 &= A, \\ m^2+n^2-e^2n^2 &= B, \\ mne^2 &= -H, \end{aligned} \tag{3}$$

while the second group is

$$\begin{aligned} \alpha(m^2+n^2)+e^2mp &= -C \\ \beta(m^2+n^2)+e^2np &= -D \\ (\alpha^2+\beta^2)(m^2+n^2)-e^2p^2 &= E. \end{aligned} \tag{4}$$

It is permissible to change signs through (2) in considering its reducibility to form (1). Thus we have to consider the solvability of equations (3) and (4) either as they stand, or with the signs of all right members changed.

3. We shall limit the discussion to the case $H \neq 0$, the special case $H=0$ being easily disposed of. With the assumption $H \neq 0$ we have $e \neq 0$, $m \neq 0$, $n \neq 0$. The sign of mn, and hence that of m/n, is opposite to that of H. Putting $m/n = k$ we find from (3),

$$Hk^2-(A-B)k-H=0. \tag{5}$$

The roots of (5) are always real and of opposite sign, so that k is uniquely determined.

The case $A=B$ is somewhat special. From (3) it is seen that

$$m^2=n^2, \quad m^2e^2=|H|, \quad m^2(2-e^2)=A,$$

so that

$$e^2=\frac{2|H|}{A+|H|}, \text{ if } A=B. \tag{6}$$

Only one selection of signs through the equation will make e^2 as given by (6) positive if $H^2-A^2 \leqslant 0$. But if $H^2-A^2=H^2-AB>0$, a change of sign through the equation will not make $e^2<0$, though such a change will alter the value of e^2 unless $A=B=0$. A criterion to fix the signs so as to render e^2 unique is considered later.

It is readily found that

$$e^2=\frac{(1+k^2)(A-B)}{A-k^2B}, \text{ if } A \neq B. \tag{7}$$

We must investigate whether e^2 as given by (7) is positive. From (5)

$$k^2-1=\frac{(A-B)k}{H}=\kappa(B-A), \quad \kappa=-k/H, \tag{8}$$

where $\kappa>0$. From (7),

$$\frac{1+k^2}{e^2}=\frac{A-k^2B}{A-B}=1+(1-k^2)\frac{B}{A-B}.$$

Using (8) we find

$$e^2=\frac{1+k^2}{1+\kappa B}. \tag{9}$$

The value of e^2 will be positive if $B>0$, and may be positive if $B<0$. Let the signs in equation (2) be changed. The other root of 5 must be used; that is, we must replace k by $-1/k$ and k by

$$-\frac{-1/k}{-H}=-\frac{1}{kH}=\frac{\kappa}{k^2},$$

and accordingly we have

$$e^2=\frac{1+k^2}{k^2-\kappa B}. \tag{10}$$

The value of e^2 in (10) will be positive if $B<0$. We see, however, that although there will always be an acceptable value for e^2, there may apparently be two such values, and it is necessary to investigate the conditions under which this is true.

The ratio of the two values of e^2 is

$$R(e^2)=\frac{1+1/k^2}{1+k^2}\cdot\frac{1-(k/H)B}{1+(1/kH)B}$$

$$=\frac{1}{k}\cdot\frac{H-kB}{Hk+B}=\frac{H/k-B}{Hk+B},$$

From (5) $$H/k=Hk-(A-B),$$

so that $$R(e^2)=\frac{Hk-A}{Hk+B}=\frac{2Hk-2A}{2Hk+2B}.$$

Using the value of k, we have

$$R(e^2)\ \frac{\pm\sqrt{(A-B)^2+4H^2}-(A+B)}{\pm\sqrt{(A-B)^2+4H^2}+(A+B)}$$

$$=\frac{[\pm\sqrt{(A-B)^2+4H^2}-(A+B)]^2}{4(H^2-AB)}.$$

Consequently the two values of e^2 will be positve if, and only if, $H^2-AB>0$. It has already been shown that an ambiguity in this case must be dealt with if $A=B$.

Being assured that we can always make $e^2>0$ we observe that we can show easily from (3) that

$$H^2-AB=(m^2+n^2)^2(e^2-1), \tag{11}$$

so that $e>$, $=$, or <1, according as $H^2-AB>$, $=$, or <0.

Having found k and e^2 we obtain m and n from the relations

$$m^2=-kH/e^2, \quad n=m/k, \tag{12}$$

taking $m>0$. (It will be observed presently that changing the signs of m and n will also change the sign of p, and such changes have no significance.)

4. We have next to consider the question of the determination of α, β, and p from equations (4). The elimination of α and β will give a quadratic in p. The question whether the quadratic will have real roots is fundamental. Since α and β are determined uniquely if p is known, it is seen that there will be two values for each of these quantities if there are two values for p. We thus have the possibility of reducing (2) to the form (1) in two ways, in one way, or in no way.

It is to be recalled that the signs of m and n are not unique, though the sign of the ratio m/n is. Changing signs of m and n merely changes the sign of p, as is seen by inspecting the first two equations in (4).

To find the quadratic for p let

$$K=m^2+n^2,$$

then from (4)

$$\alpha K+e^2mp=-C,$$

$$\beta K+e^2np=-D,$$

$$(\alpha^2+\beta^2)K-e^2p^2=E.$$

The elimination of α and β gives

$$(C+e^2mp)^2+(D+e^2np)^2-KE-Ke^2p^2=0, \quad \text{or}$$

$$(13)\quad e^2K(e^2-1)p^2+2e^2(mC+nD)p+C^2+D^2-KE=0.$$

Let Δ be the discriminant of (13); we find

$$\Delta/e^2=C^2[e^2m^2-K(e^2-1)]+D^2[e^2n^2-K(e^2-1)]$$
$$+2mne^2CD+K^2E(e^2-1).$$

Using (3), (11) and the definition of K we obtain

$$(14)\qquad \Delta/e^2=C^2B+D^2A-2CDH+(H^2-AB)E$$
$$=-\begin{vmatrix} A & H & C \\ H & B & D \\ C & D & E \end{vmatrix}.$$

5. Assume $\Delta/e^2\neq 0$. If the quantity is negative, it can be rendered positive by a change of signs through equation (2). The signs having been fixed so as to make $\Delta/e^2>0$, it follows from what has gone before that:

(1) If $H^2-AB>0$, the equation (2) is reducible to the form (1) in two ways, for e^2 will be positive.

(2) If $H^2-AB=0$, the equation (2) is reducible to the form (1) in one way, since $e=1$ and (13) is linear. (The coefficient of p cannot vanish since we are assuming $\Delta/e^2\neq 0$.)

(3) If $H^2-AB<0$, the equation (2) is reducible to the form (1) in two ways or in no way. In the latter case e^2 as given by (9) is negative and the roots of (13) are complex.

When the reduction can be made in two ways there are unique values for e, m, and n (except as to signs of m and n, as observed before), but there will be two values for p, α, and β. Thus a conic with $e\neq 1$ has two foci and two parallel directrices.

Upon eliminating p between the first two equations of (4) we find

$$\alpha n-\beta m+\frac{Cn-Dm}{m^2+n^2}=0.$$

This same equation holds for (α',β'), the second focus, and using the two equations we obtain

$$\frac{\beta-\beta'}{\alpha-\alpha'}=\frac{n}{m},$$

so that the line joining the two foci is perpendicular to the two directrices.

6. If $\Delta/e^2=0$ equation (2) factors, as is well known from purely algebraic considerations. Although the character of the locus is known, it is of interest to observe whether (2) can still be written in the form (1).

If $H^2-AB>0$ reduction to form (2) is possible in two ways with two acceptable values for e. For example

$$6x^2+24xy-26y^2=0$$

can be written

$$x^2+y^2=4\frac{(3y-x)^2}{10}$$

and by changing signs it can be written

$$x^2+y^2=\frac{4}{3}\cdot\frac{(3x+y)^2}{10}.$$

If $H^2-AB=0$, equation (13) is satisfied by no value of p (and reduction of (2) to form (1) is impossible) unless $e^2+D^2-KE=0$, in which case p is arbitrary.

As an example, consider

$$4x^2-12xy+9y^2=0.$$

We obtain $k=3/2$, $e=1$, $m=3$, $n=2$, $\alpha=-3p/13$, $\beta=-2p/13$, and the given equation can be written

$$\left(x+\frac{3}{13}p\right)^2+\left(y+\frac{2}{13}p\right)^2=\frac{(3x+2y+p)^2}{13}.$$

If $H^2-AB<0$ the factors of (2) are imaginary. Nevertheless (2) can be written in form (1), since the signs can be selected so as to make $e^2>1$. As an example consider

$$6x^2-4xy+9y^2=0.$$

We find $k=2$, $e^2=\frac{1}{2}$, $n^2=2$, $m^2=8$, $\alpha=\beta=p=0$, and the equation can be written

$$x^2+y^2=\frac{1}{2}\cdot\frac{2(2x+y)^2}{10}.$$

Since the last equation states that (x,y) is nearer to (0,0) than to the line $2\sqrt{2}x+\sqrt{2}y=0$, the only solution is $x=0$, $y=0$.*

7. Consider the question of finding the second directrix and focus when one directrix and focus are given.

*The condition that (α, β) lies on $mx+ny+p=0$ is $m\alpha+n\beta+p=0$. Using (4) we obtain

$$p=\frac{mC+nD}{(1-e^2)(m^2+n^2)}.$$

But by inspection of (13) this is the value of p when (13) has equal roots. In all the degenerate cases the focus is on the directrix and conversely.

If p_1 and p_2 are the roots of (13), we have

$$p_1+p_2=\frac{-2(mC+nD)}{(m^2+n^2)(e^2-1)} .$$

Suppose p_1 is known, and eliminate C and D by the first two equations of (4). We find

$$p_1+p_2=\frac{2(\alpha m+\beta n)}{e^2-1}+\frac{2e^2p_1}{e^2-1} ,$$

so that

$$p_2=\frac{2\alpha m+2\beta n+p_1(1+e^2)}{e^2-1} . \tag{15}$$

Let (α_1,β_1) and (α_2,β_2) be the two foci. If we use the fact that the first equation of (4) is satisfied by α_1, p_1, and by α_2, p_2, then the fact that the second equation is satisfied by β_1, p_1 and by β_2, p_2, we find

$$\begin{aligned} \alpha_2&=\alpha_1-\frac{e^2m(p_2-p_1)}{m^2+n^2} , \\ \beta_2&=\beta_1-\frac{e^2n(p_2-p_1)}{m^2+n^2} . \end{aligned} \tag{16}$$

As an example find the second focus and directrix of the ellipse with eccentricity $1/\sqrt{2}$ and $x-y-4=0$ for one directrix and $(3, -2)$ for one focus. From (15) we find $p^2=-8$, and from (16) $\alpha_2=4$, $\beta_2=-3$. Both the equations

$$(x-3)^2+(y+2)^2=\frac{1}{2}\quad\frac{(x-y-4)^2}{2}$$

and

$$(x-4)^2+(y+3)^2=\frac{1}{2}\quad\frac{(x-y-8)^2}{2}$$

reduce to

$$3x^2+3y^2+2xy-16x+8y+36=0.$$

The center of the ellipse can be easily found and also the semi-major axis a, if we assume that the distance between the directrices is $2a/e$.

Mathematical World News

Edited by
L. J. ADAMS

Professor W. L. Ayres, University of Michigan, has been appointed head of the department of mathematics at Purdue University.

The Allegheny Mountain section of the Mathematical Association of America is scheduled to meet at Washington, Pennsylvania, on October 25, 1941. The Michigan section is scheduled to meet at Detroit on November 15, 1941. The Philadelphia section is scheduled to meet at Swarthmore on November 29, 1941.

The June-July number of the *American Mathematical Monthly* included, as a supplement, a report on Industrial Mathematics, by Thornton C. Fry, Mathematical Research Director, Bell Telephone Laboratories. This thirty-eight page survey is timely and interesting, and is well worth a careful reading by mathematicians.

The popular *Outline of the History of Mathematics* by Professor R. C. Archibald has been published again, in a new and revised edition, by the Mathematical Association of America.

The William Lowell Putnam Scholarship for 1941 for study at Harvard University has been awarded to Richard F. Arens of the University of California at Los Angeles.

The Mathematical Association of America announces the publication of the sixth Carus Monograph. The book is *Fourier Series and Orthogonal Polynomials* and is by Professor Dunham Jackson, University of Minnesota.

The annual summer meeting of the M. A. A. was scheduled for September 1-4, 1941 at the University of Chicago. The meeting was in conjunction with meetings of the American Mathematical Society, the Institute of Mathematical Statistics and the Econometric Society. General lectures scheduled were:

1. *Abstract Spaces.* Professor M. H. Stone, Harvard University.
2. *Analytic Number Theory.* Professor H. A. Rademacher, University of Pennsylvania.
3. *The Calculus of Variations.* Professor G. A. Bliss, University of Chicago.
4. *Topology.* Professor Solomon Lefschetz, Princeton University.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, L. S. U., Baton Rouge, Louisiana.

SUBSCRIPTION AWARD

Awards of a year's subscription to this Magazine are made to Paul D. Thomas and to R. V. Sweeney for the best Proposal and Solution respectively, submitted by a student and printed in Volume XV. This is in accord with the Notice on page 39 of the October 1940 issue. The contest is hereby renewed for the present volume.

SOLUTIONS

No. 392. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given the circumcenter O and the excenters O_a and O_b, construct the triangle ABC.

Solution by *Geraldine Rice*, University of Oklahoma.

The midpoint M of the segment O_aO_b lies on the circumcircle of ABC. Hence OM is the circumradius and the circumcircle may be drawn. The common chord of the circumcircle and the circle described on O_aO_b as diameter is the side AB of the required triangle ABC.* The other intersection of O_aO_b with the circumcircle is the third vertex C. A similar problem is No. 134, page 80, *College Geometry* by N. A. Court.

Also solved by *Alienne Akers* (Oklahoma undergraduate), *W. B. Clarke*, *Paul D. Thomas*, and the *Proposer*.

*Since ABC is the pedal triangle of $O_aO_bO_c$, O_aA is perpendicular to O_bO_c and O_bB is perpendicular to O_aO_c. Thus the points O_a, B, A, and O_b are concyclic.—ED.

No. 393. Proposed by *Dewey C. Duncan*, Los Angeles City College.

Let the proper fraction a/b be reduced to a repeating decimal fraction whose period P consists of n digits. State and prove the general theorem of which the following are special cases:

(a) If $n=1$, then 3 is a divisor of b.
(b) If $n>1$ and 3 is not a divisor of b, then 3 is a divisor of P.
(c) If $n=3$ and 37 is not a divisor of b, then 37 is a divisor of P.

Solution by *C. W. Trigg*, Los Angeles City College.

Since every rational number can be written as a repeating decimal, there exist integers P and Q (P having n digits and Q, m digits) such that

$$\begin{aligned} a/b &= 10^{-m}Q+P(10^{-m-n}+10^{-m-2n}+10^{-m-3n}+\cdots) \\ &= 10^{-m}[Q+P(10^{-n}+10^{-2n}+10^{-3n}+\cdots)] \\ &= 10^{-m}[Q+P/(10^n-1)]. \end{aligned}$$

This relation may be written as

$$(1) \qquad (10^m a-bQ)(10^n-1)=9(111\cdots1)(10^m a-bQ)=bP.$$

The following observations are then self-evident. Each of the integers b and P is divisible by 3 or one of them is divisible by 9 (as in (b)). Each prime factor of (10^n-1) is a divisor of b or of P. Thus if n is even (so that 10^n-1 is divisible by $10^2-1=9\cdot11$), then 11 is a divisor of b or of P; if n is a multiple of 3 (so that 10^n-1 is divisible by $10^3-1=9\cdot3\cdot37$), then 37 is a divisor of b or of P(as in (c)), also if 3 is not a divisor of one of them, 3^3 is a divisor of the other.

In the Proposer's (a) it was assumed that $P\neq9$, for if $P=9$ the repeating decimal can be given a finite decimal representation, e. g. $.349999\cdots=.35=7/20$.

No. 394. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

If P, Q are points on $x^2/a^2+y^2/b^2=1$ whose eccentric angles θ_1 and θ_2 satisfy the relation $(1/2)\sin(\theta_1+\theta_2)=\cos\frac{1}{2}(\theta_1-\theta_2)$, prove that PQ envelopes the curve $x^{2/3}/a^{2/3}+y^{2/3}/b^{2/3}=1$.

Solution by *C. W. Trigg*, Los Angeles City College.

The points are P : ($a\cos\theta_1$, $b\sin\theta_1$) and Q : ($a\cos\theta_2$, $b\sin\theta_2$), so the line PQ is

$$\frac{x}{a}\cdot\cos\tfrac{1}{2}(\theta_1+\theta_2)+\frac{y}{b}\cdot\sin\tfrac{1}{2}(\theta_1+\theta_2)=\cos\tfrac{1}{2}(\theta_1-\theta_2).$$

From the given relation it follows that

$$\frac{x}{a}\cdot\cos\tfrac{1}{2}(\theta_1+\theta_2)+\frac{y}{b}\cdot\sin\tfrac{1}{2}(\theta_1+\theta_2)$$

$$=\sin\tfrac{1}{2}(\theta_1+\theta_2)\cdot\cos\tfrac{1}{2}(\theta_1+\theta_2).$$

Let $\frac{1}{2}(\theta_1+\theta_2)=\alpha$, then

$$F=\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha-\sin\alpha\cos\alpha=0. \tag{1}$$

$$\frac{\partial F}{\partial\alpha}=-\frac{x}{a}\sin\alpha+\frac{y}{b}\cos\alpha+\sin^2\alpha-\cos^2\alpha=0. \tag{2}$$

Multiply (1) by $\sin\alpha$ and (2) by $\cos\alpha$ and add the resulting equations. Then we have $\cos\alpha=y^{1/3}/b^{1/3}$.

Multiply (1) by $\cos\alpha$ and (2) by $\sin\alpha$ and subtract the resulting equations. Then we have $\sin\alpha=x^{1/3}/a^{1/3}$.

When these values are substituted in (1), we have the equation of the envelope of PQ,

$$x^{2/3}/a^{2/3}+y^{2/3}/b^{2/3}=1.$$

Also solved by *C. N. Mills* and the *Proposer*.

No. 396. Proposed by *N. A. Court*, University of Oklahoma.

The six orthogonal projections of a vertex of a tetrahedron upon the internal and external bisecting planes of the three dihedral angles which the face opposite the vertex considered makes with the remaining faces, are coplanar.

Solution by the *Proposer*.

Let M be the foot of the perpendicular from the vertex A of the tetrahedron $ABCD$ upon a bisecting plane of the dihedral angle having BC for edge, and let L be the trace of AM in the plane BCD. The lines AML and BC being rectangular, a plane may be drawn through AML perpendicular to BC and meeting BC, say, in E.

The line EM bisects the angle AEL and is perpendicular to the line AML, hence $AM=ML$. Thus the point M lies in the plane parallel to BCD and which passes through the mid-points of the edges AB, AC, AD.

Similarly for the projections of A upon the other five bisecting planes. Hence the proposition.

Note 1. That M is the mid-point of AL follows also from the fact that the perpendicular AML to one of the bisecting planes relative to BC is parallel to the other bisecting plane, and that the two bisecting planes are harmonic to the planes BCA, BCD.

Note 2. The corresponding proposition in the plane is known. See, for instance, the proposer's *College Geometry*, p. 66, Art. 95.

No. 399. Proposed by *Nelson Robinson*, Louisiana State University.

A and B are playing a game at a round table using an unlimited number of flat discs having equal radii. Each player, in turn, places a disc flat against the top of the table. After a disc is once placed on the table, its position cannot be altered. The winner is the player who places a disc in the last available space on the table. In order to win, should A move first or second and what should be his method of play?

Solution by *Edwin Comfort*, University of Arkansas.

Player A can always win by playing first, and placing his first disc in the center of the table. His subsequent moves are to place his discs in positions which are symmetric, with respect to the center of the table, to the disc last played by player B. It is evident that if player B can make a move, then player A can make the symmetric move. As a consequence player A will always make the last move and so win the game.

Also solved by the *Proposer*.

No. 402. Proposed by *E. P. Starke*, Rutgers University.

Three integral squares are to be in harmonic progression: find the smallest set. What is the general solution?

Partial solution by *D. C. Binneweg*, student, Colgate University.

Let the three integers be x, y and z $(x<y<z)$. If their squares are in harmonic progression we have $1/z^2=2/y^2-1/x^2$ or

$$z=xy/(2x^2-y^2)^{1/2}. \tag{1}$$

Solutions may be obtained from integers which satisfy $2x^2-y^2=1$ with $z=xy$. Since all solutions of this Pellian equation are furnished by the relation

$$y+x\sqrt{2}=(1+\sqrt{2})^{2n+1}, \text{ for } n=0, 1, 2, \cdots,$$

we have at once infinitely many solutions of the proposed problem. The smallest (disregarding the trivial solution for $n=0$) is $x=5$, $y=7$. $z=35$, corresponding to $n=1$.

Editor's Note. $x=5$, $y=7$, $z=35$ is indeed the smallest solution (easily seen because, in (1), $(2x^2-y^2)$ is not a square for $x=1, 2, 3, 4$, unless $x=y=z$) but the method is not entirely general (e. g. it does not furnish the solution, $x=91$, $y=119$, $z=221$.) It is not difficult to find a general solution of (1), but the present problem is an easy corrolary of No. 326. (This Magazine, March, 1940). Consider the sets (A, B, C) and (BC, AC, AB). If either forms an arithmetic progression, the other is harmonic; if either consists of relatively prime integers, so does the other. In No. 326 it was shown that the squares of

$$A=(2mn+m^2-n^2), \quad B=(m^2+n^2), \quad C=(2mn-m^2+n^2)$$

are in arithmetic progression, and are relatively prime when m, n are relatively prime and not both odd. Hence the general solution of the present problem in relatively prime integers is given by

$$x=(m^2+n^2)(2mn-m^2+n^2), \qquad y=(2mn+m^2-n^2)(2mn-m^2+n^2),$$
$$z=(m^2+n^2)(2mn+m^2-n^2).$$

Of course these may be multiplied through by an arbitrary integer if relatively prime solutions are not desired.

No. 403. Proposed by *W. Raymond Crosier*, student, Colgate University.

Consider the one-parameter family of curves given by the equation

$$y=x^3-3x^2+kx+k,$$

where the parameter k is restricted so that there exist points of maxima and minima. Find the equation of the locus of these points.

Solution by *Paul D. Thomas*, Norman, Oklahoma.

From the given equation, $y'=3x^2-6x+k=0$ for maxima and minima points. Solving

$$k=6x-3x^2.$$

This value of k placed in the given equation gives the cubic

$$y=2(3x-x^3)$$

which is the locus of the maxima and minima points of the given family of curves.

Also solved by *C. W. Trigg* and the *Proposer* who note that $x=1$ is the locus of the flex points of the family.

No. 404. Proposed by *E. P. Starke*, Rutgers University.

Compute $$\sum_{n=2}^{\infty} \frac{1}{n^2-1} .$$

Solution by *Paul D. Thomas*, Norman, Oklahoma.

Consider the following reductions:

$$\sum_{n=2}^{m} 1/(n^2-1) = \sum_{n=2}^{m} 1/2(n-1) - \sum_{n=2}^{m} 1/2(n+1)$$

$$= \sum_{n=0}^{m-2} 1/2(n+1) - \sum_{n=2}^{m} 1/2(n+1)$$

$$= \tfrac{1}{2}+\tfrac{1}{4}-1/2m-1/2(m+1),$$

in which the other terms of the two summations cancel. In the limit, as $m\to\infty$, this gives the desired result, $\frac{3}{4}$.

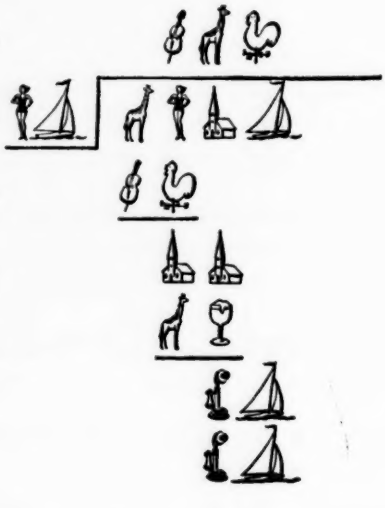
In his solution, *E. C. Kennedy* points out that this is a special case of

$$\sum_{n=a+1}^{\infty} \frac{1}{n^2-a^2} = \frac{1}{2a} \sum_{n=1}^{2a} \frac{1}{n} .$$

Also solved by *D. C. Binneweg, Edwin Comfort, A. B. Farnell, W. N. Huff, J. F. Price* and *C. W. Trigg.*

No. 406. Proposed by *Comptometer*, Chicago.

Identify the symbols in the accompanying long division example.



Solution by *M. S. Heyman*, New York City.

The procedure is simplified by using letters instead of pictures:

Bather—*A*	Weather-cock—*E*
Sailboat—*B*	Viol—*F*
Giraffe—*C*	Glass—*G*
Church—*D*	Telephone—*H*.

The first two multiplications determine A as 1, and exhibit two multiples of B (besides itself) which are less than 10. Thus $B=2$ or 3. The relation $E \cdot B = 10x+B$ determines B as 2, and E as 6. Thus the divisor is 12. Then C and F are less than 5 and not 1, 2, or 0. The subtractions show $C=F+1$, $D=C+1$, whence $F=3$, $C=4$, $D=5$, and the dividend is 4152. The remaining symbols are obvious now: $H=7$, $G=8$. The argument shows that this is a unique solution, unless an integer is represented by more than one symbol.

This problem which originated in the advertising department of *Felt and Tarrant Mfg. Co.*, was also solved by *W. B. Clarke, A. C. Cohen, Jr., Edwin Comfort, Leland Locke, Mary McCluskey, Leon Shenfil, Jean A. Thomas, C. W. Trigg, Marion C. Wicht,* and *Lucille G. Meyer.*

PROPOSALS

No. 423. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

In triangle ABC, side AB is fixed in size and position and angle C is constant. Determine the locus of the feet of the internal and external bisectors of angles A and B.

No. 424. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{a \sin x \sinh y + b \sinh 2y}{a \cos x \cosh y + b \sin 2x}.$$

No. 425. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given circles $O(R)$ and $O'(r)$, A a fixed point on circle O, B a fixed point on circle O'. Construct parallel chords AC and BD in circles O and O' respectively so that $(AC)(BD)$ is a maximum. A solution without the use of the Calculus is desired.

No. 426. Proposed by *E. P. Starke*, Rutgers University.

P is a point equidistant from two perpendicular lines, m and n. Find the locus of a point whose distance from P equals the sum of its distances from m and n.

No. 427. Proposed by *Isabella Burdick*, student, Missouri State Teachers College.

Find the length of the spiral $r\theta = a$ from (r_1,θ_1) to (r_2,θ_2).

No. 428. Proposed by *H. C. D. McClusky*, Colgate University.

An open box is to be formed from a rectangular piece of cardboard, a by b, by cutting out squares of side x at each corner and folding up the flaps. Find sets of integers, a and b, such that x is a rational number when the volume is a maximum.

No. 429. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Construct a triangle given the radius of the nine-point circle and the lengths of an internal bisector and an exsymmedian issued from the same vertex.

Defense Mathematics

is

PRACTICAL---OBJECTIVE

is

TAUGHT WITH FIELD AND LABORATORY PROJECTS

The Mathematics of the Army, the Navy, the Air Corps, as well as Industry, is alive, vivid, interesting, because the figures and formulas are based on actual situations. Many teachers have made General Mathematics, Algebra, Geometry and Trigonometry both useful and interesting to their pupils by adding laboratory and field work.

Mathematics taught with life situations is the only effective method of reaching the "slow moving" pupil. It helps him to develop concepts in quantitative thinking. The bright pupil is also benefited Mathematics becomes a living and useful thing, and the pupil is stimulated to greater effort.

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Bibliography and Reviews

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H. A. SIMMONS and JOHN W. CELL

Basic Geometry. By George David Birkhoff and Ralph Beatley. Scott, Foresman and Company, Chicago, 1941. 294 pages.

The principal feature of *Basic Geometry* is the new system which it presents, a segment of the path from counting to calculus. The combination of number and figure is a powerful tool. The authors believe that trimming the underbrush of details to a minimum will make the system of thought more apparent to the student. Their aim is to prepare students for work in advanced and modern mathematics without wading through the mass of details found in the traditional course. And truly, an able student, completing the course successfully, would have a substantial foundation for solid geometry, trigonometry, analytic geometry, vector analysis, and modern geometry.

The five fundamental postulates are:

"1. The points on any straight line can be numbered so that number differences measure distances.

"2. There is one and only one straight line through two given points.

"3. All half-lines having the same end-point can be numbered so that number differences measure angles.

"4. All straight angles have the same measure.

"5. Two triangles are similar if an angle of one equals an angle of the other and the sides including these angles are proportional."

The book provides many opportunities for the student to apply the system of logic to non-mathematical material, thus encouraging transfer of training to extra-scholastic life. The exercises are carefully graded and highly developmental.

While many terms usually defined in plane geometry texts can be placed among the undefined terms, the development of lively concepts for these terms cannot be omitted, e. g., corresponding parts. Not only are the mathematical subtleties "slurred over" as stated by the authors, but some of the definitions are carelessly stated.

There is a tendency in the book to side-step commonly used words of more than two syllables, and immediately following to employ more difficult terms. Similarly, there is a tendency to define very simple terms and depend on notions from previous work for more difficult ones. For example, on one page (66) is a precise definition of perimeter while exactly opposite it (page 67) is this comment" 'Proportion' is used without definition, its meaning being assumed from arithmetic". It has been this reviewer's experience that in an average sophomore class, for every student with a functioning concept of proportion there are at least ten students with a working notion of perimeter. The use of half-line instead of ray or vector is unfortunate.

For definitions and propositions this writer prefers bold face type to modern italics, and believes that numbered paragraphs in a mathematics text book greatly

facilitate rapid reference. However, *Basic Geometry* is an attractive book with large, clear type, many good diagrams, and interesting pictures. Every plane geometry teacher should examine this text carefully.

University High School H. TERRY.
University of Illinois

Elements of the Differential and Integral Calculus. Revised Edition. By William Anthony Granville, Percy F. Smith, and William Raymond Langley. Ginn and Co., Boston, 1941. xi+556 pages. $3.60.

To many of us who have studied and taught "Granville's", this revised edition comes as a surprise. At first superficial glance one wonders what could be done to improve a text which has proven and established itself over a long period of time. Actually, the presentation and general make-up have not been radically changed.

The authors have included in this revised edition a comprehensive chapter on hyperbolic functions and their applications. These functions are introduced through their exponential definitions, and the whole analysis is developed from this point of view. One article is devoted to the relations of these functions to the equilateral hyperbola. It is, perhaps, regrettable that the expansion of the hyperbolic functions by Maclaurin's series is left to the student in a set of exercises, and in consequence this topic is rather undeveloped. Applications to transmission lines and to mapping are considered in two articles. The entire chapter is written with characteristic clearness, and is an integral part of the text.

The chapter on multiple integrals has been improved by the addition of an article on cylindrical coordinates and their application to volumes. The chapter on differentials is slightly revised. The worded problems throughout the text have not been changed, but the drill exercises have been revised for the most part.

There are two outstanding improvements in terminology and notation: the use of the expression "prove the limit of" rather than "evaluate" when referring to indeterminate forms, and the use of ln N instead of log N when the natural logarithm of N is intended.

For conciseness and general clearness this text is hard to beat, and it should be able to continue to hold its own in an over crowded text-book field.

North Carolina State College. H. M. NAHIKIAN.

Fundamentals of Mathematics. By Moses Richardson. The Macmillan Company New York, 1941. xviii+525 pages. $3.25.

Fundamentals of Mathematics is an introduction to the vast expanse of mathematics, designed to replace for the students of arts and social sciences the "customary freshman course with emphasis on further memorized and regurgitated techniques." In each field particular emphasis is placed on the postulational development, though the proofs of a number of theorems are necessarily omitted from the text. There is considerably more than the average class, meeting three times a week, could com plete in the usual school year.

The first chapter makes an effort to allay the fears of the student and gives some advice on the reading of mathematics. The title of the second chapter, *Logic, Mathematics and Science,* perhaps best descr bes it. The next three chapters are devoted to a development of algebra as an abstract mathematical science, and it is pointed out

that each definition or assumption is really suggested by our experience with numbers. The titles of the remaining chapters are: *The Algebra of Logic and Other Algebras; Arithmetic, Exponents, and Logarithms; Impossibilities and Unsolved Problems; Analytic Geometry; Functions; Limits and the Calculus; Trigonometric Functions; Probability and Statistics; Natural Numbers and Mathematical Induction; Cardinal Numbers; Euclidean and Non-Euclidean Geometry; Two Simple Mathematical Sciences* (Groups and a Miniature Geometry); *The Nature of Mathematics.* There appears to be ample exercises to give the student the necessary practice. Bits of history and some facts about many of the great men of mathematics are inserted at appropriate intervals. For those who wish to proceed with mathematics some essential topics from algebra and trigonometry are discussed briefly in an appendix.

This book seems to have been prepared with unusual care and, as the chapter headings indicate, hardly belongs in the category of unified mathematics. It is rather the applications of a single idea, the postulational development, to the various branches of mathematics. This would surely prove a valuable text for those students able to handle it, but this reviewer fears that students, either unwilling or unable to handle the ordinary freshman work, would hardly be found reveling in the postulational development of anything.

The typographical errors observed are scarcely worth mentioning, but the enclosing of *threeness* in quotations (p. 404) seems questionable inasmuch as it is a standard English word. Also it was noticed that the *Number of the Beast* (666) is capitalized on page 281, but not on page 119. The first word of the footnote on page 415 is evidently *it*, not *if*.

The chapter of the 15 pages on famous impossible and unsolved problems will surely command the interest of the student, and the handling of logarithms, particularly the discussion of characteristic, is exceptional. The chapter on analytic geometry and that on the calculus appear in places bulky and somewhat awkward due to the fact that these chapters precede that on trigonometry. In the chapter on trigonometry the functions are first defined for an acute angle in terms of the sides of a right triangle, and to this reviewer this approach seems pedagogically undesirable.

There is a serious error in using the term *satisfy* repeatedly (pp. 134, 136, 138, etc.) before the formal definition is finally given (p. 233). The author seems to forget part of his own teaching when he says in the footnote (p. 415), "It follows from this definition that any statement about parallel lines meeting at infinity or any other forsaken place is sheer nonsense of the most self-contradictory sort." Surely the author would not complain about the definitions and postulates of anyone else, if they are consistent!

Though a certain amount of humor, or amusing statements, is certainly not out of place, the calling of a complex number a "hybrid beast" and such a statement as "hardening of the arteries is not prerequisite for excellent work in mathematics, although in the tragic case of Galois, rigor mortis was a prerequisite for recognition," hardly contribute to the scholarly tenor of the book.

University of Alabama. BRENT CLARK.

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